INSTRUCTOR'S SOLUTIONS MANUAL

DISCRETE AND COMBINATORIAL MATHEMATICS

FIFTH EDITION

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PEARSON
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Dedicated to
the memory of
Nellie and Glen (Fuzzy) Shidler

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PART 4

PART 1

FUNDAMENTALS

OF

DISCRETE MATHEMATICS

CHAPTER 1
FUNDAMENTAL PRINCIPLES OF COUNTING

Sections 1.1 and 1.2

1. (a) By the rule of sum, there are $8 + 5 = 13$ possibilities for the eventual winner.
(b) Since there are eight Republicans and five Democrats, by the rule of product we have $8 \times 5 = 40$ possible pairs of opposing candidates.
(c) The rule of sum in part (a); the rule of product in part (b).

2. By the rule of product there are $5 \times 5 \times 5 \times 5 \times 5 \times 5 = 5^6$ license plates where the first two symbols are vowels and the last four are even digits.

3. By the rule of product there are (a) $4 \times 12 \times 3 \times 2 = 288$ distinct Buicks that can be manufactured. Of these, (b) $4 \times 1 \times 3 \times 2 = 24$ are blue.

4. (a) From the rule of product there are $10 \times 9 \times 8 \times 7 = P(10,4) = 5040$ possible slates.
(b) (i) There are $3 \times 9 \times 8 \times 7 = 1512$ slates where a physician is nominated for president.
(ii) The number of slates with exactly one physician appearing is $4 \times [3 \times 7 \times 6 \times 5] = 2520$.
(iii) There are $7 \times 6 \times 5 \times 4 = 840$ slates where no physician is nominated for any of the four offices. Consequently, $5040 - 840 = 4200$ slates include at least one physician.

5. Based on the evidence supplied by Jennifer and Tiffany, from the rule of product we find that there are $2 \times 2 \times 1 \times 10 \times 10 \times 2 = 800$ different license plates.

6. (a) Here we are dealing with the permutations of 30 objects (the runners) taken 8 (the first eight finishing positions) at a time.. So the trophies can be awarded in $P(30,8) = 30!/22!$ ways.
(b) Roberta and Candice can finish among the top three runners in 6 ways. For each of these 6 ways, there are $P(28,6)$ ways for the other 6 finishers (in the top 8) to finish the race. By the rule of product there are $6 \cdot P(28,6)$ ways to award the trophies with these two runners among the top three.

7. By the rule of product there are 2^9 possibilities.

8. By the rule of product there are (a) $12!$ ways to process the programs if there are no restrictions; (b) $(4!)(8!)$ ways so that the four higher priority programs are processed first; and (c) $(4!)(5!)(3!)$ ways where the four top priority programs are processed first and the three programs of least priority are processed last.

9. (a) $(14)(12) = 168$
(b) $(14)(12)(6)(18) = 18{,}144$
(c) $(8)(18)(6)(3)(14)(12)(14)(12) = 73{,}156{,}608$

10. Consider one such arrangement – say we have three books on one shelf and 12 on the other. This can be accomplished in 15! ways. In fact for any subdivision (resulting in two nonempty shelves) of the 15 books we get 15! ways to arrange the books on the two shelves. Since there are 14 ways to subdivide the books so that each shelf has at least one book, the total number of ways in which Pamela can arrange her books in this manner is (14)(15!).

11. (a) There are four roads from town A to town B and three roads from town B to town C, so by the rule of product there are $4 \times 3 = 12$ roads from A to C that pass through B. Since there are two roads from A to C directly, there are $12 + 2 = 14$ ways in which Linda can make the trip from A to C.
(b) Using the result from part (a), together with the rule of product, we find that there are $14 \times 14 = 196$ different round trips (from A to C and back to A).
(c) Here there are $14 \times 13 = 182$ round trips.

12. (1) a,c,t (2) a,t,c (3) c,a,t (4) c,t,a (5) t,a,c (6) t,c,a

13. (a) $8! = P(8,8)$ (b) $7!$ $6!$

14. (a) $P(7,2) = 7!/(7-2)! = 7!/5! = (7)(6) = 42$
(b) $P(8,4) = 8!/(8-4)! = 8!/4! = (8)(7)(6)(5) = 1680$
(c) $P(10,7) = 10!/(10-7)! = 10!/3! = (10)(9)(8)(7)(6)(5)(4) = 604,800$
(d) $P(12,3) = 12!/(12-3)! = 12!/9! = (12)(11)(10) = 1320$

15. Here we must place a,b,c,d in the positions denoted by x: e x e x e x e x e. By the rule of product there are 4! ways to do this.

16. (a) With repetitions allowed there are 40^{25} distinct messages.
(b) By the rule of product there are $40 \times 30 \times 30 \times \ldots \times 30 \times 30 \times 40 = (40^2)(30^{23})$ messages.

17. Class A: $(2^7 - 2)(2^{24} - 2) = 2,113,928,964$
Class B: $2^{14}(2^{16} - 2) = 1,073,709,056$
Class C: $2^{21}(2^8 - 2) = 532,676,608$

18. From the rule of product we find that there are $(7)(4)(3)(6) = 504$ ways for Morgan to configure her low-end computer system.

19. (a) $7! = 5040$ (b) $4 \times 3 \times 3 \times 2 \times 2 \times 1 \times 1 = (4!)(3!) = 144$
(c) $(3!)(5)(4!) = 720$ (d) $(3!)(4!)(2) = 288$

20. (a) Since there are three A's, there are $8!/3! = 6720$ arrangements.

(b) Here we arrange the six symbols D,T,G,R,M, AAA in $6! = 720$ ways.

21. (a) $12!/(3!2!2!2!)$
(b) $[11!/(3!2!2!2!)]$ (for AG) + $[11!/(3!2!2!2!)]$ (for GA)
(c) Consider one case where all the vowels are adjacent: S,C,L,G,C,L, OIOOIA. These seven symbols can be arranged in $(7!)/(2!2!)$ ways. Since O,O,O,I,I,A can be arranged in $(6!)/(3!2!)$ ways, the number of arrangements with all the vowels adjacent is $[7!/(2!2!)][6!/(3!2!)]$.

22. (Case 1: The leading digit is 5) $(6!)/(2!)$
(Case 2: The leading digit is 6) $(6!)/(2!)^2$
(Case 3: The leading digit is 7) $(6!)/(2!)^2$
In total there are $[(6!)/(2!)][1 + (1/2) + (1/2)] = 6! = 720$ such positive integers n.

23. Here the solution is the number of ways we can arrange 12 objects — 4 of the first type, 3 of the second, 2 of the third, and 3 of the fourth. There are $12!/(4!3!2!3!) = 277,200$ ways.

24. $P(n+1,r) = (n+1)!/(n+1-r)! = [(n+1)/(n+1-r)] \cdot [n!/(n-r)!] = [(n+1)/(n+1-r)]P(n,r)$.

25. (a) $n = 10$ (b) $n = 5$
(c) $2n!/(n-2)! + 50 = (2n)!/(2n-2)! \Longrightarrow 2n(n-1) + 50 = (2n)(2n-1) \Longrightarrow n^2 = 25 \Longrightarrow n = 5$.

26. Any such path from (0,0) to (7,7) or from (2,7) to (9,14) is an arrangement of 7 R's and 7 U's. There are $(14!)/(7!7!)$ such arrangements.
In general, for m, n nonnegative integers, and any real numbers a, b, the number of such paths from (a, b) to $(a+m, b+n)$ is $(m+n)!/(m!n!)$.

27. (a) Each path consists of 2 H's, 1 V, and 7 A's. There are $10!/(2!1!7!)$ ways to arrange these 10 letters and this is the number of paths.
(b) $10!/(2!1!7!)$
(c) If a, b, and c are any real numbers and m, n, and p are nonnegative integers, then the number of paths from (a, b, c) to $(a+m, b+n, c+p)$ is $(m+n+p)!/(m!n!p!)$.

28. (a) The **for** loop for i is executed 12 times, while those for j and k are executed $10-5+1 = 6$ and $15 - 8 + 1 = 8$ times, respectively. Consequently, following the execution of the given program segment, the value of *counter* is

$$0 + 12(1) + 6(2) + 8(3) = 48.$$

(b) Here we have three tasks — T_1, T_2, and T_3. Task T_1 takes place each time we traverse the instructions in the i loop. Similarly, tasks T_2 and T_3 take place during each iteration of the j and k loops, respectively. The final value for the integer variable counter follows by the rule of sum.

29. (a) & (b) By the rule of product the **print** statement is executed $12 \times 6 \times 8 = 576$ times.

30. (a) For five letters there are $26 \times 26 \times 26 \times 1 \times 1 = 26^3$ palindromes. There are $26 \times 26 \times 26 \times 1 \times 1 \times 1 = 26^3$ palindromes for six letters.
(b) When letters may not appear more than two times, there are $26 \times 25 \times 24 = 15,600$ palindromes for either five or six letters.

31. By the rule of product there are (a) $9 \times 9 \times 8 \times 7 \times 6 \times 5 = 136,080$ six-digit integers with no leading zeros and no repeated digit. (b) When digits may be repeated there are 9×10^5 such six-digit integers.
(i) (a) $(9 \times 8 \times 7 \times 6 \times 5 \times 1)$ (for the integers ending in 0) + $(8 \times 8 \times 7 \times 6 \times 5 \times 4)$ (for the integers ending in 2,4,6, or 8) = 68,800. (b) When the digits may be repeated there are $9 \times 10 \times 10 \times 10 \times 10 \times 5 = 450,000$ six-digit even integers.
(ii) (a) $(9 \times 8 \times 7 \times 6 \times 5 \times 1)$ (for the integers ending in 0) + $(8 \times 8 \times 7 \times 6 \times 5 \times 1)$ (for the integers ending in 5) = 28,560. (b) $9 \times 10 \times 10 \times 10 \times 10 \times 2 = 180,000$.
(iii) We use the fact that an integer is divisible by 4 if and only if the integer formed by the last two digits is divisible by 4. (a) $(8 \times 7 \times 6 \times 5 \times 6)$ (last two digits are 04, 08, 20, 40, 60, or 80) + $(7 \times 7 \times 6 \times 5 \times 16)$ (last two digits are 12, 16, 24, 28, 32, 36, 48, 52, 56, 64, 68, 72, 76, 84, 92, or 96) = 33,600. (b) $9 \times 10 \times 10 \times 10 \times 25 = 225,000$.

32. (a) For positive integers n, k, where $n = 3k$, $n!/(3!)^k$ is the number of ways to arrange the n objects $x_1, x_1, x_1, x_2, x_2, x_2, \ldots, x_k, x_k, x_k$. This must be an integer.
(b) If n, k are positive integers with $n = mk$, then $n!/(m!)^k$ is an integer.

33. (a) With 2 choices per question there are $2^{10} = 1024$ ways to answer the examination.
(b) Now there are 3 choices per question and 3^{10} ways.

34. $(4!/2!)$ (No 7's) + $(4!)$ (One 7 and one 3) + $(2)(4!/2!)$ (One 7 and two 3's) + $(4!/2!)$ (Two 7's and no 3's) + $(2)(4!/2!)$ (Two 7's and one 3) + $(4!/(2!2!))$ (Two 7's and two 3's). The total gives us 102 such four-digit integers.

35. (a) 6! (b) Let A,B denote the two people who insist on sitting next to each other. Then there are 5! (A to the right of B) + 5! (B to the right of A) = 2(5!) seating arrangements.

36. (a) Locate A. There are two cases to consider. (1) There is a person to the left of A on the same side of the table. There are 7! such seating arrangements. (2) There is a person to the right of A on the same side of the table. This gives 7! more arrangements. So there are 2(7!) possibilities. (b) 7200

37. We can select the 10 people to be seated at the table for 10 in $\binom{16}{10}$ ways. For each such selection there are 9! ways of arranging the 10 people around the table. The remaining six people can be seated around the other table in 5! ways. Consequently, there are $\binom{16}{10}9!5!$ ways to seat the 16 people around the two given tables.

38. The nine women can be situated around the table in 8! ways. Each such arrangement provides nine spaces (between women) where a man can be placed. We can select six of these places and situate a man in each of them in $\binom{9}{6}6! = 9 \cdot 8 \cdot 7 \cdot 6 \cdot 5 \cdot 4$ ways. Consequently, the number of seating arrangements under the given conditions is $(8!)\binom{9}{6}6! = 2,438,553,600$.

39.

```
procedure SumOfFact(i, sum: positive integers; j,k: nonnegative integers;
                    factorial: array [0..9] of ten positive integers)
begin
   factorial [0] := 1
   for i := 1 to 9 do
      factorial [i] := i * factorial [i - 1]

   for i := 1 to 9 do
      for j := 0 to 9 do
         for k := 0 to 9 do
            begin
               sum := factorial [i] + factorial [j] + factorial [k]
               if (100 * i + 10 * j + k) = sum then
                  print (100 * i + 10 * j + k)
            end
end
```

The unique answer is 145 since $(1!) + (4!) + (5!) = 1 + 24 + 120 = 145$.

Section 1.3

1. $\binom{6}{2} = 6!/[2!(6-2)!] = 6!/(2!4!) = (6)(5)/2 = 15$

a	b	b	c	c	e
a	c	b	d	c	f
a	d	b	e	d	e
a	e	b	f	d	f
a	f	c	d	e	f

2. Order is not relevant here and Diane can make her selection in $\binom{12}{5} = 792$ ways.

3. (a) $C(10,4) = 10!/(4!6!) = (10)(9)(8)(7)/(4)(3)(2)(1) = 210$
(b) $\binom{12}{7} = 12!/(7!5!) = (12)(11)(10)(9)(8)/(5)(4)(3)(2)(1) = 792$

(c) $C(14,12) = 14!/(12!2!) = (14)(13)/(2)(1) = 91$
(d) $\binom{15}{10} = 15!/(10!5!) = (15)(14)(13)(12)(11)/(5)(4)(3)(2)(1) = 3003$

4. (a) $2^6 - 1 = 63$ (b) $\binom{6}{3} = 20$ (c) $\binom{6}{2} + \binom{6}{4} + \binom{6}{6} = 31$

5. (a) There are $P(5,3) = 5!/(5-3)! = 5!/2! = (5)(4)(3) = 60$ permutations of size 3 for the five letters m, r, a, f, and t.
(b) There are $C(5,3) = 5!/[3!(5-3)!] = 5!/(3!2!) = 10$ combinations of size 3 for the five letters m, r, a, f, and t. They are

a,f,m	a,f,r	a,f,t	a,m,r	a,m,t
a,r,t	f,m,r	f,m,t	f,r,t	m,r,t

6.

$$\binom{n}{2} + \binom{n-1}{2} = (\frac{1}{2})(n)(n-1) + (\frac{1}{2})(n-1)(n-2) = (\frac{1}{2})(n-1)[n+(n-2)] =$$

$(\frac{1}{2})(n-1)(2n-2) = (n-1)^2$.

7. (a) $\binom{20}{12}$ (b) $\binom{10}{6}\binom{10}{6}$
(c) $\binom{10}{2}\binom{10}{10}$(2 women)$+\binom{10}{4}\binom{10}{8}$ (4 women)$+\ldots+\binom{10}{10}\binom{10}{2}$ (10 women) $= \sum_{i=1}^{5}\binom{10}{2i}\binom{10}{12-2i}$
(d) $\binom{10}{7}\binom{10}{5}$ (7 women) $+ \binom{10}{8}\binom{10}{4}$ (8 women) $+ \binom{10}{9}\binom{10}{3}$ (9 women) $+$
$\binom{10}{10}\binom{10}{2}$ (10 women) $= \sum_{i=7}^{10}\binom{10}{i}\binom{10}{12-i}$.
(e) $\sum_{i=8}^{10}\binom{10}{i}\binom{10}{12-i}$

8. (a) $\binom{4}{1}\binom{13}{5}$ (b) $\binom{4}{4}\binom{48}{1}$ (c) $\binom{13}{1}\binom{4}{4}\binom{48}{1}$ (d) $\binom{4}{3}\binom{4}{2}$
(e) $\binom{4}{3}\binom{12}{1}\binom{4}{2}$ (f) $\binom{13}{1}\binom{4}{3}\binom{12}{1}\binom{4}{2} = 3744$
(g) $\binom{13}{1}\binom{4}{3}\binom{48}{1}\binom{44}{1}/2$ (Division by 2 is needed since no distinction is made for the order in which the other two cards are drawn.) This result equals $54,912 = \binom{13}{1}\binom{4}{3}\binom{48}{2} - 3744 =$ $\binom{13}{1}\binom{4}{3}\binom{12}{2}\binom{4}{1}\binom{4}{1}$.
(h) $\binom{13}{2}\binom{4}{2}\binom{4}{2}\binom{44}{1}$.

9. (a) $\binom{8}{2}$ (b) $\binom{8}{4}$ (c) $\binom{8}{6}$ (d) $\binom{8}{6} + \binom{8}{7} + \binom{8}{8}$.

10. $\binom{12}{5}$; $\binom{10}{3}$.

11. (a) $\binom{10}{7} = 120$ (b) $\binom{8}{5} = 56$ (c) $\binom{6}{4}\binom{4}{3}$ (four of the first six) $+\binom{6}{5}\binom{4}{2}$ (five of the first six) $+\binom{6}{6}\binom{4}{1}$ (all of the first six) $= (15)(4) + (6)(6) + (1)(4) = 100$.

12. (a) The first three books can be selected in $\binom{12}{3}$ ways. The next three in $\binom{9}{3}$ ways. The third set of three in $\binom{6}{3}$ ways and the fourth set in $\binom{3}{3}$ ways. Consequently, the 12 books can be distributed in $\binom{12}{3}\binom{9}{3}\binom{6}{3}\binom{3}{3} = (12!)/[(3!)^4]$ ways.

(b) $\binom{12}{4}\binom{8}{4}\binom{4}{2}\binom{2}{2} = (12!)/[(4!)^2(2!)^2]$.

13. The letters M,I,I,I,P,P,I can be arranged in $[7!/(4!)(2!)]$ ways. Each arrangement provides eight locations (one at the start of the arrangement, one at the finish, and six between letters) for placing the four nonconsecutive S's. Four of these locations can be selected in $\binom{8}{4}$ ways. Hence, the total number of these arrangements is $\binom{8}{4}$ $[7!/(4!)(2!)]$.

14. $\binom{n}{11} = 12,376$ when $n = 17$.

15. (a) Two distinct points determine a line. With 15 points, no three collinear, there are $\binom{15}{2}$ possible lines.

(b) There are $\binom{25}{3}$ possible triangles or planes, and $\binom{25}{4}$ possible tetrahedra.

16. (a) $\sum_{i=1}^{6}(i^2+1) = (1^2+1)+(2^2+1)+(3^2+1)+(4^2+1)+(5^2+1)+(6^2+1) = 2+5+10+17+26+37$
$= 97$

(b) $\sum_{j=-2}^{2}(j^3-1) = [(-2)^3-1]+[(-1)^3-1]+(0^3-1)+(1^3-1)+(2^3-1) = -9-2-1+0+7$
$= -5$

(c) $\sum_{i=0}^{10}[1+(-1)^i] = 2+0+2+0+2+0+2+0+2+0+2 = 12$

(d) $\sum_{k=n}^{2n}(-1)^k = [(-1)^n+(-1)^{n+1}]+[(-1)^{n+2}+(-1)^{n+3}]+\ldots+[(-1)^{2n-1}+(-1)^{2n}]$
$= 0+0+\ldots+0 = 0$

(e) $\sum_{i=1}^{6}i(-1)^i = -1+2-3+4-5+6 = 3$

17. (a) $\sum_{k=2}^{n}\frac{1}{k!}$ (b) $\sum_{i=1}^{7}i^2$ (c) $\sum_{j=1}^{7}(-1)^{j-1}j^3 = \sum_{k=1}^{7}(-1)^{k+1}k^3$

(d) $\sum_{i=0}^{n}\frac{i+1}{n+i}$ (e) $\sum_{i=0}^{n}(-1)^i\left[\frac{n+i}{(2i)!}\right]$

18. (a) $10!/(4!3!3!)$ (b) $\binom{10}{8}2^2+\binom{10}{9}2+\binom{10}{10}$

(c) $\binom{10}{4}$ (four 1's, six 0's) + $\binom{10}{2}\binom{8}{1}$ (two 1's, one 2, seven 0's) +
$\binom{10}{2}$ (two 2's, eight 0's)

19. $\binom{10}{3}$ (three 1's, seven 0's) + $\binom{10}{1}\binom{9}{1}$ (one 1, one 2, eight 0's) +
$\binom{10}{1}$ (one 3, nine 0's) = 220
$\binom{10}{4}+\binom{10}{2}+\binom{10}{1}\binom{9}{2}+\binom{10}{1}\binom{9}{1} = 705$

$(2^{10})(\sum_{i=0}^{5}\binom{10}{2i})$ – Select an even number of locations for 0,2. This is done in $\binom{10}{2i}$ ways for $0 \le i \le 5$. Then for the $2i$ positions selected there are two choices; for the $10-2i$ remaining positions there are also two choices – namely, 1,3.

20. (a) We can select 3 vertices from A, B, C, D, E, F, G, H in $\binom{8}{3}$ ways, so there are $\binom{8}{3} = 56$ distinct inscribed triangles.
(b) $\binom{8}{4} = 70$ quadrilaterals.
(c) The total number of polygons is $\binom{8}{3}+\binom{8}{4}+\binom{8}{5}+\binom{8}{6}+\binom{8}{7}+\binom{8}{8} = 2^8-[\binom{8}{0}+\binom{8}{1}+\binom{8}{2}] = 256-[1+8+28] = 219$.

21. There are $\binom{n}{3}$ triangles if sides of the n-gon may be used. Of these $\binom{n}{3}$ triangles, when $n \ge 4$ there are n triangles that use two sides of the n-gon and $n(n-4)$ triangles that use only one side. So if the sides of the n-gon cannot be used, then there are $\binom{n}{3} - n - n(n-4)$, $n \ge 4$, triangles.

22. (a) From the rule of product it follows that there are $4 \times 4 \times 6 = 96$ terms in the complete expansion of $(a+b+c+d)(e+f+g+h)(u+v+w+x+y+z)$.
(b) The terms bvx and egu do not occur as summands in this expansion.

23. (a) $\binom{12}{9}$ (b) $\binom{12}{9}(2^3)$
(c) Let $a = 2x$ and $b = -3y$. By the binomial theorem the coefficient of a^9b^3 in the expansion of $(a+b)^{12}$ is $\binom{12}{9}$. But $\binom{12}{9}\, a^9b^3 = \binom{12}{9}(2x)^9(-3y)^3 = \binom{12}{9}(2^9)(-3)^3\, x^9y^3$, so the coefficient of x^9y^3 is $\binom{12}{9}(2^9)(-3)^3$.

24. $\binom{n}{n_1}\binom{n-n_1}{n_2}\binom{n-n_1-n_2}{n_3}\cdots\binom{n-n_1-n_2-\ldots-n_{t-1}}{n_t} =$
$\left(\frac{n!}{n_1!(n-n_1)!}\right)\left(\frac{(n-n_1)!}{n_2!(n-n_1-n_2)!}\right)\left(\frac{(n-n_1-n_2)!}{n_3!(n-n_1-n_2-n_3)!}\right)\cdots\left(\frac{n_t!}{n_t!0!}\right)$
$= \frac{n!}{n_1!n_2!n_3!\ldots n_t!}$.

25. (a) $\binom{4}{1,1,2} = 12$ (b) $\binom{4}{0,1,1,2} = 12$
(c) $\binom{4}{1,1,2}(2)(-1)(-1)^2 = -24$ (d) $\binom{4}{1,1,2}(-2)(3)^2 = -216$
(e) $\binom{8}{3,2,1,2}(2)^3(-1)^2(3)(-2)^2 = 161,280$

26. (a) $\binom{10}{2,2,2,2,2} = (10!)/(2!)^5 = 113,400$
(b) $\binom{12}{2,2,2,2,4}(2)^2(-1)^2(3)^2(1)^2(-2)^4 = [(12!)/[(2!)^4(4!)]](2)^2(3)^2(2)^4 = 718,502,400$
(c) $\binom{12}{0,2,2,2,2,4}(1)^2(-2)^2(1)^2(5)^2(3)^4 = [(12!)/(0!)(2!)^4(4!)]](2)^2(5)^2(3)^4 = 10,103,940,000$

27. In each of parts (a)–(e) replace the variables by 1 and evaluate the results.
(a) 2^3 (b) 2^{10} (c) 3^{10} (d) 4^5 (e) 4^{10}

28. a) $\sum_{i=0}^{n} \frac{1}{i!(n-i)!} = \frac{1}{n!}\sum_{i=0}^{n} \frac{n!}{i!(n-i)!} = \frac{1}{n!}\sum_{i=0}^{n}\binom{n}{i} = 2^n/n!$

b) $\sum_{i=0}^{n} \frac{(-1)^i}{i!(n-i)!} = \frac{1}{n!}\sum_{i=0}^{n} \frac{(-1)^i n!}{i!(n-i)!} = \frac{1}{n!}\sum_{i=0}^{n}(-1)^i\binom{n}{i} = \frac{1}{n!}(0) = 0.$

29. $n\binom{m+n}{m} = n\frac{(m+n)!}{m!n!} = \frac{(m+n)!}{m!(n-1)!} =$
$(m+1)\frac{(m+n)!}{(m+1)(m!)(n-1)!} = (m+1)\frac{(m+n)!}{(m+1)!(n-1)!} = (m+1)\binom{m+n}{m+1}$

30. The sum is the binomial expansion of $(1+2)^n = 3^n$.

31. (a) $1 = [(1+x)-x]^n = (1+x)^n - \binom{n}{1}x^1(1+x)^{n-1} + \binom{n}{2}x^2(1+x)^{n-2} - \ldots + (-1)^n\binom{n}{n}x^n$.
(b) $1 = [(2+x)-(x+1)]^n$ (c) $2^n = [(2+x)-x]^n$

32. $\sum_{i=0}^{50}\binom{50}{i}8^i = (1+8)^{50} = 9^{50} = [(\pm 3)^2)]^{50} = (\pm 3)^{100}$, so $x = \pm 3$.

33. (a) $\sum_{i=1}^{3}(a_i - a_{i-1}) = (a_1 - a_0) + (a_2 - a_1) + (a_3 - a_2) = a_3 - a_0$

(b) $\sum_{i=1}^{n}(a_i - a_{i-1}) = (a_1 - a_0) + (a_2 - a_1) + (a_3 - a_2) + \ldots + (a_{n-1} - a_{n-2}) + (a_n - a_{n-1}) =$
$a_n - a_0$

(c) $\sum_{i=1}^{100}(\frac{1}{i+2} - \frac{1}{i+1}) = (\frac{1}{3} - \frac{1}{2}) + (\frac{1}{4} - \frac{1}{3}) + (\frac{1}{5} - \frac{1}{4}) + \ldots + (\frac{1}{101} - \frac{1}{100}) + (\frac{1}{102} - \frac{1}{101}) =$
$\frac{1}{102} - \frac{1}{2} = \frac{1-51}{102} = \frac{-50}{102} = \frac{-25}{51}$.

34.

```
procedure Select2(i,j: positive integers)
begin
   for i := 1 to 5 do
      for j := i + 1 to 6 do
         print (i,j)
end
```

```
procedure Select3(i,j,k: positive integers)
begin
   for i := 1 to 4 do
      for j := i + 1 to 5 do
         for k := j + 1 to 6 do
            print (i,j,k)
end
```

Section 1.4

1. Let $x_i, 1 \le i \le 5$, denote the amounts given to the five children.
(a) The number of integer solutions of $x_1 + x_2 + x_3 + x_4 + x_5 = 10,\ 0 \le x_i,\ 1 \le i \le 5$, is $\binom{5+10-1}{10} = \binom{14}{10}$. Here $n = 5,\ r = 10$.
(b) Giving each child one dime results in the equation $x_1 + x_2 + x_3 + x_4 + x_5 = 5,\ 0 \le x_i,\ 1 \le i \le 5$. There are $\binom{5+5-1}{5} = \binom{9}{5}$ ways to distribute the remaining five dimes.
(c) Let x_5 denote the amount for the oldest child. The number of solutions to $x_1 + x_2 + x_3 + x_4 + x_5 = 10,\ 0 \le x_i,\ 1 \le i \le 4,\ 2 \le x_5$ is the number of solutions to $y_1 + y_2 + y_3 + y_4 + y_5 = 8,\ 0 \le y_i,\ 1 \le i \le 5$, which is $\binom{5+8-1}{8} = \binom{12}{8}$.

2. Let $x_i,\ 1 \le i \le 5$, denote the number of candy bars for the five children with x_1 the number for the youngest. $(x_1 = 1):\ x_2 + x_3 + x_4 + x_5 = 14$. Here there are $\binom{4+14-1}{14} = \binom{17}{14}$ distributions. $(x_1 = 2):\ x_2 + x_3 + x_4 + x_5 = 13$. Here the number of distributions is $\binom{4+13-1}{13} = \binom{16}{13}$. The answer is $\binom{17}{14} + \binom{16}{13}$ by the rule of sum.

3. $\binom{4+20-1}{20} = \binom{23}{20}$

4. (a) $\binom{31}{12}$ (b) $\binom{31+12-1}{12} = \binom{42}{12}$
(c) There are 31 ways to have 12 cones with the same flavor. So there are $\binom{42}{12} - 31$ ways to order the 12 cones and have at least two flavors.

5. (a) 2^5
(b) For each of the n distinct objects there are two choices. If an object is not selected, then one of the n identical objects is used in the selection. This results in 2^n possible selections of size n.

6. $\binom{12}{4,4,4}\binom{22}{12}$

7. (a) $\binom{4+32-1}{32} = \binom{35}{32}$ (b) $\binom{4+28-1}{28} = \binom{31}{28}$
(c) $\binom{4+8-1}{8} = \binom{11}{8}$ (d) 1
(e) $x_1 + x_2 + x_3 + x_4 = 32,\ x_i \ge -2,\ 1 \le i \le 4$. Let $y_i = x_i + 2,\ 1 \le i \le 4$. The number of solutions to the given problem is then the same as the number of solutions to $y_1 + y_2 + y_3 + y_4 = 40,\ y_i \ge 0,\ 1 \le i \le 4$. This is $\binom{4+40-1}{40} = \binom{43}{40}$.
(f) $\binom{4+28-1}{28} - \binom{4+3-1}{3} = \binom{31}{28} - \binom{6}{3}$, where the term $\binom{6}{3}$ accounts for the solutions where $x_4 \ge 26$.

8. For the chocolate donuts there are $\binom{3+5-1}{5} = \binom{7}{5}$ distributions. There are $\binom{3+4-1}{4} = \binom{6}{4}$ ways to distribute the jelly donuts. By the rule of product there are $\binom{7}{5}\binom{6}{4}$ ways to distribute the donuts as specified.

9. $$230,230 = \binom{n+20-1}{20} = \binom{n+19}{20} \Longrightarrow n = 7$$

10. Here we want the number of integer solutions for $x_1 + x_2 + x_3 + x_4 + x_5 + x_6 = 100$, $x_i \geq 3$, $1 \leq i \leq 6$. (For $1 \leq i \leq 6$, x_i counts the number of times the face with i dots is rolled.) This is equal to the number of nonnegative integer solutions there are to $y_1+y_2+y_3+y_4+y_5+y_6 = 82$, $y_i \geq 0$, $1 \leq i \leq 6$. Consequently the answer is $\binom{6+82-1}{82} = \binom{87}{82}$.

11. (a) $\binom{10+5-1}{5} = \binom{14}{5}$ (b) $\binom{7+5-1}{5} + 3\binom{7+4-1}{4} + 3\binom{7+3-1}{3} + \binom{7+2-1}{2} = \binom{11}{5} + 3\binom{10}{4} + 3\binom{9}{3} + \binom{8}{2}$, where the first summand accounts for the case where none of 1,3,7 appears, the second summand for when exactly one of 1,3,7 appears once, the third summand for the case of exactly two of these digits appearing once each, and the last summand for when all three appear.

12. (a) The number of solutions for $x_1 + x_2 + \ldots + x_5 < 40, x_i \geq 0$, $1 \leq i \leq 5$, is the same as the number for $x_1 + x_2 + \ldots + x_5 \leq 39, x_i \geq 0$, $1 \leq i \leq 5$, and this equals the number of solutions for $x_1 + x_2 + \ldots + x_5 + x_6 = 39, x_i \geq 0$, $1 \leq i \leq 6$. There are $\binom{6+39-1}{39} = \binom{44}{39}$ such solutions.
(b) Let $y_i = x_i + 3$, $1 \leq i \leq 5$, and consider the inequality $y_1 + y_2 + \ldots + y_5 \leq 54$, $y_i \geq 0$. There are [as in part (a)] $\binom{6+54-1}{54} = \binom{59}{54}$ solutions.

13. (a) $\binom{4+4-1}{4} = \binom{7}{4}$.
(b) $\binom{3+7-1}{7}$ (container 4 has one marble) $+\binom{3+5-1}{5}$ (container 4 has three marbles) $+\binom{3+3-1}{3}$ (container 4 has five marbles) $+\binom{3+1-1}{1}$ (container 4 has seven marbles) $= \sum_{i=0}^{3} \binom{9-2i}{7-2i}$.

14. (a) $\binom{8}{2,4,1,0,1}(3)^2(2)^4$
(b) The terms in the expansion have the form $v^a w^b x^c y^d z^e$ where a, b, c, d, e are nonnegative integers that sum to 8. There are $\binom{5+8-1}{8} = \binom{12}{8}$ terms.

15. Consider one such distribution – the one where there are six books on each of the four shelves. Here there are 24! ways for this to happen. And we see that there are also 24! ways to place the books for any other such distribution.

The number of distributions is the number of positive integer solutions to

$$x_1 + x_2 + x_3 + x_4 = 24.$$

This is the same as the number of nonnegative integer solutions for

$$y_1 + y_2 + y_3 + y_4 = 20.$$

[Here $y_i + 1 = x_i$ for all $1 \leq i \leq 4$.]
So there are $\binom{4+20-1}{20} = \binom{23}{20}$ such distributions of the books, and consequently, $\binom{23}{20}(24!)$ ways in which Beth can arrange the 24 books on the four shelves with at least one book on each shelf.

16. For equation (1) we need the number of nonnegative integer solutions for $w_1 + w_2 + w_3 + \ldots + w_{19} = n - 19$, where $w_i \geq 0$ for all $1 \leq i \leq 19$. This is $\binom{19+(n-19)-1}{(n-19)} = \binom{n-1}{n-19}$. The number of positive integer solutions for equation (2) is the number of nonnegative integer solutions for

$$z_1 + z_2 + z_3 + \ldots + z_{64} = n - 64,$$

and this is $\binom{64+(n-64)-1}{(n-64)} = \binom{n-1}{n-64}$.
So $\binom{n-1}{n-19} = \binom{n-1}{n-64} = \binom{n-1}{63}$ and $n - 19 = 63$. Hence $n = 82$.

17. (a) $\binom{5+12-1}{12} = \binom{16}{12}$ (b) 5^{12}

18. (a) There are $\binom{3+6-1}{6} = \binom{8}{6}$ solutions for $x_1+x_2+x_3 = 6$ and $\binom{4+31-1}{31} = \binom{34}{31}$ solutions for $x_4 + x_5 + x_6 + x_7 = 31$, where $x_i \geq 0$, $1 \leq i \leq 7$. By the rule of product the pair of equations has $\binom{8}{6}\binom{34}{31}$ solutions.
(b) $\binom{5}{3}\binom{34}{31}$

19. Here there are $r = 4$ nested **for** loops, so $1 \leq m \leq k \leq j \leq i \leq 20$. We are making selections, with repetition, of size $r = 4$ from a collection of size $n = 20$. Hence the **print** statement is executed $\binom{20+4-1}{4} = \binom{23}{4}$ times.

20. Here there are $r = 3$ nested **for** loops and $1 \leq i \leq j \leq k \leq 15$. So we are making selections, with repetition, of size $r = 3$ from a collection of size $n = 15$. Therefore the statement

$$counter := counter + 1$$

is executed $\binom{15+3-1}{3} = \binom{17}{3}$ times, and the final value of the variable *counter* is $10 + \binom{17}{3} = 690$.

21. The **begin-end** segment is executed $\binom{10+3-1}{3} = \binom{12}{3} = 220$ times. After the execution of this segment the value of the variable *sum* is $\sum_{i=1}^{220} i = (220)(221)/2 = 24,310$.

22. $\binom{n+2}{3} = \sum_{i=1}^{n} \binom{i+1}{2} \Longrightarrow \frac{(n+2)(n+1)n}{6} = \frac{1}{2}\sum_{i=1}^{n}(i+1)i \Longrightarrow \frac{(n+2)(n+1)n}{6} = \frac{1}{2}\sum_{i=1}^{n} i^2 + \frac{1}{2}\sum_{i=1}^{n} i \Longrightarrow \frac{1}{2}\sum_{i=1}^{n} i^2 = \frac{(n+2)(n+1)n}{6} - \frac{(n+1)n}{4} \Longrightarrow \sum_{i=1}^{n} i^2 = n(n+1)[\frac{n+2}{3} - \frac{1}{2}] = n(n+1)[\frac{2n+4-3}{6}] = \frac{n(n+1)(2n+1)}{6}$.

23. (a) Put one object into each container. Then there are $m - n$ identical objects to place into n distinct containers. This yields $\binom{n+(m-n)-1}{m-n} = \binom{m-1}{m-n} = \binom{m-1}{n-1}$ distributions.
(b) Place r objects into each container. The remaining $m - rn$ objects can then be distributed among the n distinct containers in $\binom{n+(m-rn)-1}{m-rn} = \binom{m-1+(1-r)n}{m-rn} = \binom{m-1+(1-r)n}{n-1}$ ways.

24. (a)

procedure *Selections1(i,j:* nonnegative integers)

```
begin
    for i := 0 to 10 do
        for j := 0 to 10 - i do
            print (i,j, 10 - i - j)
end
```

(b) For all $1 \leq i \leq 4$ let $y_i = x_i + 2 \geq 0$. Then the number of integer solutions to $x_1 + x_2 + x_3 + x_4 = 4$, where $-2 \leq x_i$ for $1 \leq i \leq 4$, is the number of integer solutions to $y_1 + y_2 + y_3 + y_4 = 12$, where $y_i \geq 0$ for $1 \leq i \leq 4$. We use this observation in the following.

```
procedure Selections2(i,j,k: nonnegative integers)
begin
    for i := 0 to 12 do
        for j := 0 to 12 - i do
            for k := 0 to 12 - i - j do
                print (i,j,k, 12 - i - j - k)
end
```

25. If the summands must all be even, then consider one such composition – say,

$$20 = 10 + 4 + 2 + 4 = 2(5 + 2 + 1 + 2).$$

Here we notice that $5 + 2 + 1 + 2$ provides a composition of 10. Further, each composition of 10, when multiplied through by 2, provides a composition of 20, where each summand is even. Consequently, we see that the number of compositions of 20, where each summand is even, equals the number of compositions of 10 – namely, $2^{10-1} = 2^9$.

26. Each such composition can be factored as k times a composition of m. Consequently, there are 2^{m-1} compositions of n, where $n = mk$ and each summand in a composition is a multiple of k.

27. a) Here we want the number of integer solutions for $x_1 + x_2 + x_3 = 12$, $x_1, x_3 > 0$, $x_2 = 7$. The number of integer solutions for $x_1 + x_3 = 5$, with $x_1, x_3 > 0$, is the same as the number of integer solutions for $y_1 + y_3 = 3$, with $y_1, y_3 \geq 0$. This is $\binom{2+3-1}{3} = \binom{4}{3} = 4$.

b) Now we must also consider the integer solutions for $w_1 + w_2 + w_3 = 12$, $w_1, w_3 > 0$, $w_2 = 5$. The number here is $\binom{2+5-1}{5} = \binom{6}{5} = 6$.

Consequently, there are $4 + 6 = 10$ arrangements that result in three runs.

c) The number of arrangements for four runs requires two cases [as above in part (b)].

If the first run consists of heads, then we need the number of integer solutions for $x_1 + x_2 + x_3 + x_4 = 12$, where $x_1 + x_3 = 5$, $x_1, x_3 > 0$ and $x_2 + x_4 = 7$, $x_2, x_4 > 0$. This number is $\binom{2+3-1}{3}\binom{2+5-1}{5} = \binom{4}{3}\binom{6}{5} = 4 \cdot 6 = 24$. When the first run consists of tails we get $\binom{6}{5}\binom{4}{3} = 6 \cdot 4 = 24$ arrangements.

In all there are $2(24) = 48$ arrangements with four runs.

d) If the first run starts with an H, then we need the number of integer solutions for $x_1 + x_2 + x_3 + x_4 + x_5 = 12$ where $x_1 + x_3 + x_5 = 5$, $x_1, x_3, x_5 > 0$ and $x_2 + x_4 = 7$, $x_2, x_4 > 0$. This is $\binom{3+2-1}{2}\binom{2+5-1}{5} = \binom{4}{2}\binom{6}{5} = 36$. For the case where the first run starts with a T, the number of arrangements is $\binom{3+4-1}{4}\binom{2+3-1}{3} = \binom{6}{4}\binom{4}{3} = 60$.

In total there are $36 + 60 = 96$ ways for these 12 tosses to determine five runs.

e) $\binom{3+4-1}{4}\binom{3+2-1}{2} = \binom{6}{4}\binom{4}{2} = 90$ – the number of arrangements which result in six runs, if the first run starts with an H. But this is also the number when the first run starts with a T. Consequently, six runs come about in $2 \cdot 90 = 180$ ways.

f) $2\binom{1+4-1}{4}\binom{1+6-1}{6}+2\binom{2+3-1}{3}\binom{2+5-1}{5}+2\binom{3+2-1}{2}\binom{3+4-1}{4}+2\binom{4+1-1}{1}\binom{4+3-1}{3}+2\binom{5+0-1}{0}\binom{5+2-1}{2} = 2\sum_{i=0}^{4}\binom{4}{4-i}\binom{6}{6-i} = 2[1 \cdot 1 + 4 \cdot 6 + 6 \cdot 15 + 4 \cdot 20 + 1 \cdot 15] = 420$.

28. (a) For $n \geq 4$, consider the strings made up of n bits – that is, a total of n 0's and 1's. In particular, consider those strings where there are (exactly) two occurrences of 01. For example, if $n = 6$ we want to include strings such as 010010 and 100101, but not 101111 or 010101. How many such strings are there?
(b) For $n \geq 6$, how many strings of n 0's and 1's contain (exactly) three occurrences of 01?
(c) Provide a combinatorial proof for the following:

$$\text{For } n \geq 1, \quad 2^n = \binom{n+1}{1} + \binom{n+1}{3} + \cdots + \begin{cases} \binom{n+1}{n}, & n \text{ odd} \\ \binom{n+1}{n+1}, & n \text{ even.} \end{cases}$$

(a) A string of this type consists of x_1 1's followed by x_2 0's followed by x_3 1's followed by x_4 0's followed by x_5 1's followed by x_6 0's, where,

$$x_1 + x_2 + x_3 + x_4 + x_5 + x_6 = n, \quad x_1, x_6 \geq 0, \quad x_2, x_3, x_4, x_5 > 0.$$

The number of solutions to this equation equals the number of solutions to

$$y_1 + y_2 + y_3 + y_4 + y_5 + y_6 = n - 4, \quad \text{where } y_i \geq 0 \text{ for } 1 \leq i \leq 6.$$

This number is $\binom{6+(n-4)-1}{n-4} = \binom{n+1}{n-4} = \binom{n+1}{5}$.

(b) For $n \geq 6$, a string with this structure has x_1 1's followed by x_2 0's followed by x_3 1's ... followed by x_8 0's, where

$$x_1 + x_2 + x_3 + \cdots + x_8 = n, \quad x_1, x_8 \geq 0, \quad x_2, x_3, \ldots, x_7 > 0.$$

The number of solutions to this equation equals the number of solutions to

$$y_1 + y_2 + y_3 + \cdots + y_8 = n - 6, \quad \text{where } y_i \geq 0 \text{ for } 1 \leq i \leq 8.$$

This number is $\binom{8+(n-6)-1}{n-6} = \binom{n+1}{n-6} = \binom{n+1}{7}$.

(c) There are 2^n strings in total and $n+1$ strings where there are k 1's followed by $n-k$ 0's, for $k = 0, 1, 2, \ldots, n$. These $n+1$ strings contain no occurrences of 01, so there are $2^n - (n+1) = 2^n - \binom{n+1}{1}$ strings that contain at least one occurrence of 01. There are $\binom{n+1}{3}$ strings that contain (exactly) one occurrence of 01, $\binom{n+1}{5}$ strings with (exactly) two occurrences, $\binom{n+1}{7}$ strings with (exactly) three occurrences, ... , and for
(i) n odd, we can have at most $\frac{n-1}{2}$ occurrences of 01. The number of strings with $\frac{n-1}{2}$ occurrences of 01 is the number of integer solutions for

$$x_1 + x_2 + \cdots + x_{n+1} = n, \ x_1, x_{n+1} \geq 0, \quad x_2, x_3, \ldots, x_n > 0.$$

This is the same as the number of integer solutions for

$$y_1 + y_2 + \cdots + y_{n+1} = n - (n-1) = 1, \ \text{ where } \ y_1, y_2, \ldots, y_{n+1} \geq 0.$$

This number is $\binom{(n+1)+1-1}{1} = \binom{n+1}{1} = \binom{n+1}{n} = \binom{n+1}{2(\frac{n-1}{2})+1}$.
(ii) n even, we can have at most $\frac{n}{2}$ occurrences of 01. The number of strings with $\frac{n}{2}$ occurrences of 01 is the number of integer solutions for

$$x_1 + x_2 + \cdots + x_{n+2} = n, \ x_1, x_{n+2} \geq 0, \quad x_2, x_3, \ldots, x_n > 0.$$

This is the same as the number of integer solutions for

$$y_1 + y_2 + \cdots + y_{n+2} = n - n = 0, \ \text{ where } \ y_i \geq 0 \ \text{ for } \ 1 \leq i \leq n+2.$$

This number is $\binom{(n+2)+0-1}{0} = \binom{n+1}{0} = \binom{n+1}{n+1} = \binom{n+1}{2(\frac{n}{2})+1}$.
Consequently,

$$2^n - \binom{n+1}{1} = \binom{n+1}{3} + \binom{n+1}{5} + \cdots + \begin{cases} \binom{n+1}{n}, & n \text{ odd} \\ \binom{n+1}{n+1}, & n \text{ even,} \end{cases}$$

and the result follows.

Section 1.5

1.

$$\binom{2n}{n} - \binom{2n}{n-1} = \frac{(2n)!}{n!n!} - \frac{(2n)!}{(n-1)!(n+1)!} =$$

$$\frac{(2n)!(n+1)}{(n+1)!n!} - \frac{(2n)!n}{n!(n+1)!} = \frac{(2n)![(n+1)-n]}{(n+1)!n!} = \frac{1}{(n+1)}\frac{(2n)!}{n!n!} =$$

$$(\frac{1}{n+1})\binom{2n}{n}$$

2. $b_7 = 429$ $b_8 = 1430$ $b_9 = 4862$ $b_{10} = 16796$

3. (a) $5(= b_3)$; $14(= b_4)$

(b) For $n \geq 0$ there are $b_n(= \frac{1}{(n+1)}\binom{2n}{n})$ such paths from (0,0) to (n, n).

(c) For $n \geq 0$ the first move is U and the last is H.

4. (a) $b_6 = 132$ (b) $b_5 = 42$ (c) $b_7 = 429$

5. Using the results in the third column of Table 1.10 we have:

111000	110010	101010
1 2 3 4 5 6	1 2 5 3 4 6	1 3 5 2 4 6

6.
(a) (i) 1 3 4 7 / 2 5 6 8 (ii) 1 2 5 7 / 3 4 6 8 (iii) 1 2 3 5 / 4 6 7 8

(b) (i) 10111000 (ii) 11100010 (iii) 11011000

7. There are $b_5(= 42)$ ways.

8. (a) (i) 1110001010 (ii) 1010101010 (iii) 1111001000

(b) (i) $(((ab(c(de \leftrightarrow (((ab)(c(de)))f)$
(ii) $((ab((cd(e \leftrightarrow ((ab)((cd)(ef)))$
(iii) $(a(((bc(de \leftrightarrow (a(((bc)(de))f))$

9. (i) When $n = 4$ there are $14(= b_4)$ such diagrams.

(ii) For any $n \geq 0$, there are b_n different drawings of n semicircles on and above a horizontal line, with no two semicircles intersecting. Consider, for instance, the diagram in part (f) of the figure. Going from left to right, write 1 the first time you encounter a semicircle and write 0 the second time that semicircle is encountered. Here we get the list 110100. The list 110010 corresponds with the drawing in part (g). This correspondence shows that the number of such drawings for n semicircles is the same as the number of lists of n 1's and n 0's where, as the list is read from left to right, the number of 0's never exceeds the number of 1's.

10. (a) In total there are $\binom{10}{7} = \binom{10}{3}$ paths from (0,0) to (7,3), each made up of seven R's and three U's. From these $\binom{10}{7}$ paths we remove those that violate the stated condition – namely, those paths where the number of U's exceeds the number R's (at some first position in the path). For example, consider one such path:

$$R\,U\,R\,U\,U\,R\,R\,R\,R\,R.$$

Here the condition is violated, for the first time, after the third U. Transform the given path as follows:

$$R\,U\,R\,U\,U\vdots R\,R\,R\,R\,R \leftrightarrow R\,U\,R\,U\,U\,U\,U\,U\,U\,U.$$

Here the entries up to and including the first violation remain unchanged, while those following the first violation are changed: R's become U's and U's become R's. This correspondence shows us that the number of paths that violate the given condition is the same as the number of paths made up of eight U's and two R's – and there are $\binom{10}{8} = \binom{10}{2}$ such paths.

Consequently, the answer is
$\binom{10}{7} - \binom{10}{8} = \frac{10!}{7!3!} - \frac{10!}{8!2!} = \frac{10!(8)}{8!3!} - \frac{10!(3)}{8!3!} = (\frac{5}{8})\frac{10!}{7!3!} = (\frac{7+1-3}{7+1})\binom{10}{7}$.

(b) $\binom{m+n}{n} - \binom{m+n}{n+1} = \frac{(m+n)!}{n!m!} - \frac{(m+n)!}{(n+1)!(m-1)!}$
$= \frac{(m+n)!(n+1)-(m+n)!m}{(n+1)!m!} = (\frac{n+1-m}{n+1})(\frac{(m+n)!}{n!m!}) = (\frac{n+1-m}{n+1})\binom{m+n}{n}$.
[Note that when $m = n$, this becomes $(\frac{1}{n+1})\binom{2n}{n}$, the formula for the nth Catalan number.]

11. Consider one of the $(\frac{1}{6+1})\binom{2\cdot 6}{6} = (\frac{1}{7})\binom{12}{6}$ ways in which the \$5 and \$10 bills can be arranged – say,

(*) \$5, \$5, \$10, \$5, \$5, \$10, \$10, \$10, \$5, \$5, \$10, \$10.

Here we consider the six \$5 bills as indistinguishable – likewise, for the six \$10 bills. However, we consider the patrons as distinct. Hence, there are 6! ways for the six patrons, each with a \$5 bill, to occupy positions 1, 2, 4, 5, 9, and 10, in the arrangement (*). Likewise, there are 6! ways to locate the other six patrons (each with a \$10 bill). Consequently, here the number of arrangements is

$$(\frac{1}{7})\binom{12}{6}(6!)(6!) = (\frac{1}{7})(12!) = 68,428,800.$$

Supplementary Exercises

1. $\binom{4}{1}\binom{7}{2} + \binom{4}{2}\binom{7}{4} + \binom{4}{3}\binom{7}{6}$

2. (a) 5^9 (b) $5(4^8)$

3.

Select any four of these twelve points (on the circumference). As seen in the figure, these points determine a pair of chords that intersect. Consequently, the largest number of points of intersection for all possible chords is $\binom{12}{4} = 495$.

4. (a) $\binom{25}{2}^3$

(b) $3\binom{25}{1}^2\binom{25}{4}$ (four hymns from one book, one from each of the other two) $+ 6\binom{25}{1}\binom{25}{2}\binom{25}{3}$ (one hymn from one book, two hymns from a second book, and three from the third book) $+\binom{25}{2}^3$ (two hymns from each of the three books).

5. (a) 10^{25}

(b) There are 10 choices for the first flag. For the second flag there are 11 choices: The nine poles with no flag, and above or below the first flag on the pole where it is situated. There are 12 choices for the third flag, 13 choices for the fourth,. . ., and 34 choices for the last (25th). Hence there are $(34!)/(9!)$ possible arrangements.

(c) There are 25! ways to arrange the flags. For each arrangement consider the 24 spaces, one between each pair of flags. Selecting 9 of these spaces provides a distribution among the 10 flagpoles where every flagpole has at least one flag and order is relevant. Hence there are $(25!)\binom{24}{9}$ such arrangements.

6. Consider the 45 heads and the 46 positions they determine: (1) One position to the left of the first head; (2) One position between the i-th head and the $(i+1)$-st head, where $1 \leq i \leq 44$; and, (3) One position to the right of the 45-th (last) head. To answer the question posed we need to select 15 of the 46 positions. This we can do in $\binom{46}{15}$ ways.

In an alternate way, let x_i denote the number of heads to the left of the i-th tail, for $1 \leq i \leq 15$. Let x_{16} denote the number of heads to the right of the 15th tail. Then we want the number of integer solutions for

$$x_1 + x_2 + x_3 + \ldots + x_{15} + x_{16} = 45,$$

where $x_1 \geq 0$, $x_{16} \geq 0$, and $x_i > 0$ for $2 \leq i \leq 15$. This is the number of integer solutions for

$$y_1 + y_2 + y_3 + \ldots + y_{15} + y_{16} = 31,$$

with $y_i \geq 0$ for $1 \leq i \leq 16$. Consequently the answer is $\binom{16+31-1}{31} = \binom{46}{31} = \binom{46}{15}$.

7. (a) $C(12, 8)$ (b) $P(12, 8)$

8. There are $(7!/2!)$ ways to arrange the seven symbols OE,W,N,N,D,R,G. In each arrangement there are 6 locations for the I so that it is not adjacent to a vowel, so there are $(6)(7!/2!)$ arrangements. The three vowels can be divided up into a pair and a single vowel in six ways (order counts), so the total number of arrangements is $(6^2)(7!/2!)$.

9. (a) There are two blocks, for example, that differ only in size. There are four that differ only in color, one that differs only in the material used for construction, and five that differ only in shape. In total there are $2 + 4 + 1 + 5 = 12$ blocks that differ from the ***small red wooden square*** block in exactly one way.

(b) There are $\binom{4}{2} = 6$ ways of selecting the two differing properties. Each such pair must be considered separately.

(i) Material, size: Here there are $1 \times 2 = 2$ such blocks.
(ii) Material, color: This pair yields $1 \times 4 = 4$ such blocks.
(iii) Material, shape: For this pair we obtain $1 \times 5 = 5$ such blocks.
(iv) Size, color: Here we get $2 \times 4 = 8$ of the blocks.
(v) Size, shape: This pair gives us $2 \times 5 = 10$ such blocks.
(vi) Color, shape: For this pair we find $4 \times 5 = 20$ of the blocks we need to count.

In total there are $2 + 4 + 5 + 8 + 10 + 20 = 49$ of Dustin's blocks that differ from the *large blue plastic hexagonal* block in exactly two ways.

10. Since 'R' is the 18th letter of the alphabet, the first and middle initials can be chosen in $\binom{17}{2} = (17)(16)/2 = 136$ ways.

Alternately, since 'R' is the 18th letter of the alphabet, consider what happens when the middle initial is any letter between 'B' and 'Q'. For middle initial 'Q' there are 16 possible first initials. For middle initial 'P' there are 15 possible choices. Continuing back to 'B' where there is only one choice (namely 'A') for the first initial, we find that the total number of choices is $1 + 2 + 3 + \ldots + 15 + 16 = (16)(17)/2 = 136$.

11. The number of linear arrangements of the 11 horses is $11!/(5!3!3!)$. Each circular arrangement represents 11 linear arrangements, so there are $(1/11)[11!/(5!3!3!)]$ ways to arrange the horses on the carousel.

12. (a) $P(16, 12)$ (b) $\binom{12}{2} P(15, 10)$

13. (a) (i) $\binom{5}{4} + \binom{5}{2}\binom{4}{2} + \binom{4}{4}$ (ii) $\binom{5+4-1}{4} + \binom{5+2-1}{2}\binom{4+2-1}{2} + \binom{4+4-1}{4} = \binom{8}{4} + \binom{6}{2}\binom{5}{2} + \binom{7}{4}$ (iii) $\binom{8}{4} + \binom{6}{2}\binom{5}{2} + \binom{7}{4} - 9$
(b) (i) $\binom{5}{1}\binom{4}{3} + \binom{5}{3}\binom{4}{1}$ (ii) and (iii) $\binom{5}{1}\binom{4+3-1}{3} + \binom{5+3-1}{3}\binom{4}{1} = \binom{5}{1}\binom{6}{3} + \binom{7}{3}\binom{4}{1}$.

14. (a) If there are no restrictions Mr. Kelly can make the assignments in $12! = 479{,}001{,}600$ ways.
(b) Mr. DiRocco and Mr. Fairbanks can be assigned in $4 \times 3 = 12$ ways, and the other 10 assistants can then be assigned in $10!$ ways. Consequently, in this situation, Mr. Kelly can make one of $12(10!) = 43{,}545{,}600$ assignments.
(c) Suppose that Mr. Hyland is assigned to the first floor and Mr. Thornhill is assigned to the third floor. This can be accomplished in $4 \times 4 \times (10!) = 58,060,800$ ways. There are $3 \times 2 = 6$ ways to assign these two assistants to different floors, so in this case we have $(3 \times 2) \times [4 \times 4 \times (10!)] = 348,364,800$ possibilities.
Alternately, from part (b), there are $3 \times [12(10!)] = 130,636,800$ ways in which Mr. Hyland and Mr. Thornhill could be assigned to the same floor — and $(12!) - [(3)(12)(10!)] = 348{,}364{,}800$.

15. (a) For each increasing four-digit integer we have four distinct digits, which can only be arranged in one way. These four digits can be chosen in $\binom{9}{4} = 126$ ways. And these same four digits can also be arranged as a decreasing four-digit integer.

To complete the solution we must account for the decreasing four-digit integers where the units digit is 0. There are $\binom{9}{3} = 84$ of these.

Consequently there are $2\binom{9}{4} + \binom{9}{3} = 343$ such four-digit integers.

(b) For each nondecreasing four-digit integer we have four nonzero digits, with repetitions allowed. These four digits can be selected in $\binom{9+4-1}{4} = \binom{12}{4}$ ways. And these same four digits account for a nonincreasing four-digit integer. So at this point we have $2\binom{12}{4} - 9$ of the four-digit integers we want to count. (The reason we subtract 9 is because we have counted the nine integers 1111, 2222, 3333, ..., 9999 twice in $2\binom{12}{4}$.)

We have not accounted for those nonincreasing four-digit integers where the units digit is 0. There are $\binom{10+3-1}{3} - 1 = \binom{12}{3} - 1$ of these four-digit integers. (Here we subtracted 1 since we do not want to include 0000.)

Therefore there are $[2\binom{12}{4} - 9] + [\binom{12}{3} - 1] = [2\binom{12}{4} + \binom{12}{3}] - 10 = 1200$ such four-digit integers.

16. (a) $\binom{5}{2,1,2}(1/2)^2(-3)^2 = 135/2$
(b) Each term is of the form $x^{n_1}y^{n_2}z^{n_3}$ where each n_i, $1 \leq i \leq 3$, is a nonnegative integer and $n_1 + n_2 + n_3 = 5$. Consequently, there are $\binom{3+5-1}{5} = \binom{7}{5}$ terms.
(c) Replace x, y, and z by 1. Then the sum of all the coefficients in the expansion is $((1/2) + 1 - 3)^5 = (-3/2)^5$.

17. (a) First place person A at the table. There are five distinguishable places available for A (e.g., any of the positions occupied by A,B,C,D,E in Fig. 1.11(a)). Then position the other nine people relative to A. This can be done in 9! ways, so there are (5)(9!) seating arrangements.
(b) There are three distinct ways to position A,B so that they are seated on longer sides of the table across from each other. The other eight people can then be located in 8! different ways, so the total number of arrangements is (3)(8!).

18. (a) For $x_1 + x_2 + x_3 = 6$ there are $\binom{3+6-1}{6} = \binom{8}{6}$ nonnegative integer solutions. With $x_1 + x_2 + x_3 = 6$ and $x_1 + x_2 + x_3 + x_4 + x_5 = 15$, the number of nonnegative integer solutions for $x_4 + x_5 = 9$ is $\binom{2+9-1}{9} = \binom{10}{9}$. The number of solutions for the pair of equations is $\binom{8}{6}\binom{10}{9}$.
(b) Let $0 \leq k \leq 6$. For $x_1 + x_2 + x_3 = k$ there are $\binom{3+k-1}{k} = \binom{k+2}{k}$ solutions. To solve $x_4 + x_5 \leq 15 - k$, consider $x_4 + x_5 + x_6 = 15 - k$, $x_4, x_5, x_6 \geq 0$. Here there are $\binom{3+15-k-1}{15-k} = \binom{17-k}{15-k}$ solutions. The total number of solutions is $\sum_{k=0}^{6}\binom{k+2}{k}\binom{17-k}{15-k}$.

19. (a) Here A must win set 5 and exactly two of the four earlier sets. This can be done in $\binom{4}{2}$ ways. With seven possible scores for each set there are $\binom{4}{2}7^5$ ways for the scores to be recorded.

(b) Here A can win in four sets in $\binom{3}{2}$ ways, and scores can be recorded in $\binom{3}{2}7^4$ ways. So if A wins in four or five sets, then the scores can be recorded in $[\binom{3}{2}7^4 + \binom{4}{2}7^5]$ ways. Since B may be the winner, the final answer is $2[\binom{3}{2}7^4 + \binom{4}{2}7^5]$.

20. We can choose r objects from n in $\binom{n}{r}$ ways. Once the r objects are selected they can be arranged in a circle in $(r-1)!$ ways. So there are $\binom{n}{r}(r-1)!$ circular arrangements of the n objects taken r at a time.

21. For every positive integer n, $0 = (1-1)^n = \binom{n}{0}(1)^0 - \binom{n}{1}(1)^1 + \binom{n}{2}(1)^2 - \binom{n}{3}(1)^3 + \ldots + (-1)^n\binom{n}{n}(1)^n$, and $\binom{n}{0} + \binom{n}{2} + \binom{n}{4} + \ldots = \binom{n}{1} + \binom{n}{3} + \binom{n}{5} + \ldots$

22. (a) $7!/3!$ (b) $5!$ (c) $\binom{5}{3}(4!)$

23. (a) There are $P(20,12) = \frac{20!}{8!} = (20)(19)(18)\cdots(11)(10)(9)$ ways in which Francesca can fill her bookshelf.
(b) There are $\binom{17}{9}$ ways in which Francesca can select nine other books. Then she can arrange those nine books and the three books on tennis on her bookshelf in $12!$ ways. Consequently, among the arrangements in part (a), there are $\binom{17}{9}(12!)$ arrangements that include Francesca's three books on tennis.

24. Following the execution of this program segment the value of *counter* is
$10+(12-1+1)(r-1+1)(2)+[3+4+\ldots+(s-3+1)](4)+(12-3+1)(6)+(t-7+1)(8) =$
$10+(12)(r)(2)+[(1/2)(s-3+1)(s-3+2)-2-1](4)+(10)(6)+(t-6)(8) =$
$22+24r+8t+2(s-2)(s-1)-12 = 14+24r+8t+2s(s-3)$.

25. (a) For 17 there must be an odd number, between 1 and 17 inclusive, of 1's.
For $2k+1$ 1's, where $0 \le k \le 8$, there are $2k+2$ locations to select, with repetitions allowed. The selection size is the number of 2's, which is $(1/2)[17-(2k+1)] = 8-k$. The selection can be made in $\binom{2k+2+(8-k)-1}{8-k} = \binom{9+k}{8-k}$ ways, and so the answer is $\sum_{k=0}^{8}\binom{9+k}{8-k} = 2584$.
(b) In the case of 18 the number of 1's must be even: $2k$, for $0 \le k \le 9$. If there are $2k$ 1's, there are $2k+1$ locations, with repetitions allowed, for the $(1/2)(18-2k) = 9-k$ 2's. The selection can be made in $\binom{2k+1+(9-k)-1}{9-k} = \binom{9+k}{9-k}$ ways, and the answer is $\sum_{k=0}^{9}\binom{9+k}{9-k} = 4181$.
(c) For n odd, let $n = 2k+1$ for $k \ge 0$. The number of ways to write n as an ordered sum of 1's and 2's is $\sum_{i=0}^{k}\binom{k+1+i}{k-i}$.
For n even, let $n = 2k$ for $k \ge 1$. Here the answer is $\sum_{i=1}^{k}\binom{k+i}{k-i}$.

26. (a) (i) 1 (one 3) + 1 (three 3's) + 1 (five 3's) = 3.
(ii) $\binom{8}{1}$ (one 3) + $\binom{7}{3}$ (three 3's) + $\binom{6}{5}$ (five 3's).
(b) (i) 1 (no 3's) + 1 (two 3's) + 1 (four 3's) + 1 (six 3's) = 4.
(ii) $\binom{9}{0}$ (no 3's) + $\binom{8}{2}$ (two 3's) + $\binom{7}{4}$ (four 3's) + $\binom{6}{6}$ (six 3's).

27. (a) The number of positive integer solutions to the given equation is the same as the number of nonnegative integer solutions for $y_1 + y_2 + \ldots + y_r = n - r$, where $y_i \geq 0$ for all $1 \leq i \leq r$. Here there are $\binom{r+(n-r)-1}{n-r} = \binom{n-1}{n-r} = \binom{n-1}{r-1}$ solutions.
(b) The total is $\sum_{r=1}^{n} \binom{n-1}{r-1} = \binom{n-1}{0} + \binom{n-1}{1} + \ldots + \binom{n-1}{n-1} = 2^{n-1}$.

28. (a) There are $5 - 1 = 4$ horizontal moves and $9 - 2 = 7$ vertical moves. One can arrange 4 R's and 7 U's in $11!/(4!7!)$ ways.
(b) Since a diagonal move takes the place of one horizontal move and one vertical move, the number of diagonal moves is between 0 and 4, inclusive. The resulting cases are as follows:

(0 D's):	4 R's,	7 U's:	$11!/(4!7!)$
(1 D's):	3 R's,	6 U's:	$10!/(1!3!6!)$
(2 D's):	2 R's,	5 U's:	$9!/(2!2!5!)$
(3 D's):	1 R,	4 U's:	$8!/(3!1!4!)$
(4 D's):	0 R's,	3 U's:	$7!/(4!0!3!)$

The answer is the sum of the results: $\sum_{i=0}^{4}[(11-i)!/(i!(4-i)!(7-i)!)]$.

29. (a) $11!/(7!4!)$
(b) $[11!/(7!4!)] - [4!/(2!2!)][4!/(3!1!)]$
(c) $[11!/(7!4!)] + [10!/(6!3!1!)] + [9!/(5!2!2!)] + [8!/(4!1!3!)] + [7!/(3!0!4!)]$ (for part (a))
$\{[11!/(7!4!)] + [10!/(6!3!1!)] + [9!/(5!2!2!)] + [8!/(4!1!3!)] + [7!/(3!0!4!)]\} -$
$[\{[4!/(2!2!)] + [3!/(1!1!1!)] + [2!/2!]\} \times \{[4!/(3!1!)] + [3!/(2!1!)]\}]$ (for part (b)).

30. Here we want certain paths from (1,1) to (14,4) where the moves are of the form:
$(m, n) \to (m+1, n+1)$, if the $(n+1)$-st ballot is for Katalin.
$(m, n) \to (m+1, n-1)$, if the $(n+1)$-st ballot is for Donna.

These paths are the ones that never touch or cross the horizontal (or x-) axis. In general, an ordered pair (m, n) here indicates that m ballots have been counted with Katalin leading by n votes. The number of ways to count the ballots according to the prescribed conditions is

$$\binom{13}{8} - \binom{13}{9} = 1287 - 715 = 572.$$

31. Each rectangle (contained within the 8×5 grid) is determined by four corners of the form $(a, b), (c, b), (c, d), (a, d)$, where a, b, c, d are integers with $0 \leq a < c \leq 8$ and $0 \leq b < d \leq 5$. We can select the pair a, c in $\binom{9}{2}$ ways and the pair b, d in $\binom{6}{2}$ ways. Consequently, the number of rectangles is $\binom{9}{2}\binom{6}{2} = 540$.

32. Here we consider the number of integer solutions for

$$x_1 + x_2 + x_3 = 6, \quad x_i > 0, \quad 1 \leq i \leq 3, \text{ and } w_1 + w_2 = 6, \quad w_i > 0, \quad 1 \leq i \leq 2.$$

This equals the number of integer solutions for

$$y_1 + y_2 + y_3 = 3, \quad y_i \geq 0, \quad 1 \leq i \leq 3, \text{ and } z_1 + z_2 = 3, \quad z_i \geq 0, \quad 1 \leq i \leq 2.$$

So the answer is $\binom{3+3-1}{3}\binom{2+3-1}{3} = \binom{5}{3}\binom{4}{3}$.

33. There are $\binom{6}{4} = 15$ ways to choose the four quarters when Hunter will take these electives. For each of these choices of four quarters, there are $12 \cdot 11 \cdot 10 \cdot 9$ ways to assign the electives. So, in total, there are $\binom{6}{4} \cdot 12 \cdot 11 \cdot 10 \cdot 9 = 178,200$ ways for Hunter to select and schedule these four electives.

34. Consider the family as one unit. Then we are trying to arrange nine distinct objects – the family and the eight other people – around the table. This can be done in $8!$ ways. Since the family unit can be arranged in four ways, the total number of arrangements under the prescribed conditions is $4(8!)$.

CHAPTER 2
FUNDAMENTALS OF LOGIC

Section 2.1

1. The sentences in parts (a), (c), (d), and (f) are statements.

2. The statements in parts (a), (c), and (f) are primitive statements.

3. Since $p \rightarrow q$ is false the truth value for p is 1 and that of q is 0. Consequently, the truth values for the given compound statements are

(a) 0 (b) 0 (c) 1 (d) 0

4. (a) $r \rightarrow q$ (b) $q \rightarrow p$ (c) $(s \wedge r) \rightarrow q$

5. (a) If triangle ABC is equilateral, then it is isosceles.
(b) If triangle ABC is not isosceles, then it is not equilateral.
(c) Triangle ABC is equilateral if and only if it is equiangular.
(d) Triangle ABC is isosceles but it is not equilateral.
(e) If triangle ABC is equiangular, then it is isosceles.

6. (a) True (1) (b) False (0) (c) True (1)

7. (a) If Darci practices her serve daily then she will have a good chance of winning the tennis tournament.
(c) If Mary is to be allowed on Larry's motorcycle, then she must wear her helmet.

8.

p	q	$p \vee q$	(a) $\neg(p \vee \neg q) \rightarrow \neg p$	$p \rightarrow q$	$q \rightarrow p$	(d) $(p \rightarrow q) \rightarrow (q \rightarrow p)$
0	0	0	1	1	1	1
0	1	1	1	1	0	0
1	0	1	1	0	1	1
1	1	1	1	1	1	1

p	q	$p \rightarrow q$	$p \wedge (p \rightarrow q)$	(e) $[p \wedge (p \rightarrow q)] \rightarrow q$	(f)	(g)
0	0	1	0	1	1	0
0	1	1	0	1	1	1
1	0	0	0	1	1	0
1	1	1	1	1	1	0

p	q	r	$q \to r$	(b) $p \to (q \to r)$	$p \to q$	(c) $(p \to q) \to r$	(h)
0	0	0	1	1	1	0	1
0	0	1	1	1	1	1	1
0	1	0	0	1	1	0	1
0	1	1	1	1	1	1	1
1	0	0	1	1	0	1	1
1	0	1	1	1	0	1	1
1	1	0	0	0	1	0	1
1	1	1	1	1	1	1	1

9. Propositions (a), (e), (f), and (h) are tautologies.

10.

p	q	r	$\overbrace{p \to (q \to r)}^{s}$	$\overbrace{(p \to q) \to (p \to r)}^{t}$	$s \to t$
0	0	0	1	1	1
0	0	1	1	1	1
0	1	0	1	1	1
0	1	1	1	1	1
1	0	0	1	1	1
1	0	1	1	1	1
1	1	0	0	0	1
1	1	1	1	1	1

11. (a) $2^5 = 32$ (b) 2^n

12. (a) $[(p \wedge q) \wedge r] \to (s \vee t)$ is false (0) when $(p \wedge q) \wedge r$ is true (1) and $s \vee t$ is false (0). Hence p, q, and r must be true (1) while s and t must be false (0).

13. $p : 0$; $r : 0$; $s : 0$

14. (a) $n = 9$ (b) $n = 19$ (c) $n = 19$

15.
(a) $m = 3,\ n = 6$ (b) $m = 3,\ n = 9$ (c) $m = 18,\ n = 9$
(d) $m = 4,\ n = 9$ (e) $m = 4,\ n = 9$

16.
(a) $10^2 - 10 = 90$ (b) $20^2 - 20 = 380$
(c) $(10)(20) - 10 = 190$ (d) $(20)(10) - 10 = 190$

17. Consider the following possibilities:
(i) Suppose that either the first or the second statement is the true one. Then statements (3) and (4) are false — so their negations are true. And we find from (3) that Tyler did not eat the piece of pie — while from (4) we conclude that Tyler did eat the pie.
(ii) Now we'll suppose that statement (3) is the only true statement. So statements (3) and (4) no longer contradict each other. But now statement (2) is false, and we have Dawn

guilty (from statement (2)) and Tyler guilty (from statement (3)).
(iii) Finally, consider the last possibility — that is, statement (4) is the true one. Once again statements (3) and (4) do not contradict each other, and here we learn from statement (2) that Dawn is the vile culprit.

Section 2.2

1. (a)

(i)

p	q	r	$q \wedge r$	$p \to (q \wedge r)$	$p \to q$	$p \to r$	$(p \to q) \wedge (p \to r)$
0	0	0	0	1	1	1	1
0	0	1	0	1	1	1	1
0	1	0	0	1	1	1	1
0	1	1	1	1	1	1	1
1	0	0	0	0	0	0	0
1	0	1	0	0	0	1	0
1	1	0	0	0	1	0	0
1	1	1	1	1	1	1	1

(ii)

p	q	r	$p \vee q$	$(p \vee q) \to r$	$p \to r$	$q \to r$	$(p \to r) \wedge (q \to r)$
0	0	0	0	1	1	1	1
0	0	1	0	1	1	1	1
0	1	0	1	0	1	0	0
0	1	1	1	1	1	1	1
1	0	0	1	0	0	1	0
1	0	1	1	1	1	1	1
1	1	0	1	0	0	0	0
1	1	1	1	1	1	1	1

(iii)

p	q	r	$q \vee r$	$p \to (q \vee r)$	$p \to q$	$\neg r \to (p \to q)$
0	0	0	0	1	1	1
0	0	1	1	1	1	1
0	1	0	1	1	1	1
0	1	1	1	1	1	1
1	0	0	0	0	0	0
1	0	1	1	1	0	1
1	1	0	1	1	1	1
1	1	1	1	1	1	1

b)

$[p \rightarrow (q \vee r)]$	$\Longleftrightarrow [\neg r \rightarrow (p \rightarrow q)]$	From part (iii) of part (a)
	$\Longleftrightarrow [\neg r \rightarrow (\neg p \vee q)]$	By the 2nd Substitution Rule, and $(p \rightarrow q) \Longleftrightarrow (\neg p \vee q)$
	$\Longleftrightarrow [\neg(\neg p \vee q) \rightarrow \neg\neg r]$	By the lst Substitution Rule, and $(s \rightarrow t) \Longleftrightarrow (\neg t \rightarrow \neg s)$, for primitive statements s, t
	$\Longleftrightarrow [(\neg\neg p \wedge \neg q) \rightarrow r]$	By DeMorgan's Law, Double Negation and the 2nd Substitution Rule
	$\Longleftrightarrow [(p \wedge \neg q) \rightarrow r]$	By Double Negation and the 2nd Substitution Rule

2.

p	q	$p \wedge q$	$p \vee (p \wedge q)$
0	0	0	0
0	1	0	0
1	0	0	1
1	1	1	1

3. a) For a primitive statement s, $s \vee \neg s \Longleftrightarrow T_0$. Replace each occurrence of s by $p \vee (q \wedge r)$ and the result follows by the 1st Substitution Rule.

b) For primitive statements s, t we have $(s \rightarrow t) \Longleftrightarrow (\neg t \rightarrow \neg s)$. Replace each occurrence of s by $p \vee q$, and each occurrence of t by r, and the result is a consequence of the 1st Substitution Rule.

4. (1) $[(p \wedge q) \wedge r] \vee [(p \wedge q) \wedge \neg r] \Longleftrightarrow (p \wedge q) \wedge (r \vee \neg r) \Longleftrightarrow (p \wedge q) \wedge T_0 \Longleftrightarrow p \wedge q$.
(2) $[(p \wedge q) \vee \neg q] \Longleftrightarrow (p \vee \neg q) \wedge (q \vee \neg q) \Longleftrightarrow (p \vee \neg q) \wedge T_0 \Longleftrightarrow p \vee \neg q$.
Therefore, the given statement simplifies to $(p \vee \neg q) \rightarrow s$ or $(q \rightarrow p) \rightarrow s$

5. a) Kelsey placed her studies before her interest in cheerleading, but she (still) did not get a good education.

b) Norma is not doing her mathematics homework or Karen is not practicing her piano lesson.

c) Harold did pass his C++ course and he did finish his data structures project, but he did not graduate at the end of the semester.

6. (a) $\neg[p \wedge (q \vee r) \wedge (\neg p \vee \neg q \vee r)] \Longleftrightarrow \neg p \vee (\neg q \wedge \neg r) \vee (p \wedge q \wedge \neg r) \Longleftrightarrow (\neg q \wedge \neg r) \vee [\neg p \vee (p \wedge q \wedge \neg r)] \Longleftrightarrow (\neg q \wedge \neg r) \vee [T_0 \wedge (\neg p \vee (q \wedge \neg r))] \Longleftrightarrow (\neg q \wedge \neg r) \vee [\neg p \vee (q \wedge \neg r)] \Longleftrightarrow \neg p \vee [(\neg q \vee q) \wedge \neg r] \Longleftrightarrow \neg p \vee \neg r$.
(b) $\neg[(p \wedge q) \rightarrow r] \Longleftrightarrow \neg[\neg(p \wedge q) \vee r] \Longleftrightarrow (p \wedge q) \wedge \neg r$.
(c) $p \wedge (q \vee \neg r)$ (d) $\neg p \wedge \neg q \wedge \neg r$

7. a)

p	q	$(\neg p \vee q) \wedge (p \wedge (p \wedge q))$	$p \wedge q$
0	0	0	0
0	1	0	0
1	0	0	0
1	1	1	1

b) $(\neg p \wedge q) \vee (p \vee (p \vee q)) \iff p \vee q$

8. (a) $q \rightarrow p \iff \neg q \vee p$, so $(q \rightarrow p)^d \iff \neg q \wedge p$.
(b) $p \rightarrow (q \wedge r) \iff \neg p \vee (q \wedge r)$, so $[p \rightarrow (q \wedge r)]^d \iff \neg p \wedge (q \vee r)$.
(c) $p \leftrightarrow q \iff (p \rightarrow q) \wedge (q \rightarrow p) \iff (\neg p \vee q) \wedge (\neg q \vee p)$, so $(p \leftrightarrow q)^d \iff (\neg p \wedge q) \vee (\neg q \wedge p)$.
(d) $p \veebar q \iff (p \wedge \neg q) \vee (\neg p \wedge q)$, so $(p \veebar q)^d \iff (p \vee \neg q) \wedge (\neg p \vee q)$.

9. (a) If $0+0=0$, then $2+2=1$.
Let $p: 0+0=0$, $q: 1+1=1$.
(The implication: $p \rightarrow q$) – If $0+0=0$, then $1+1=1$. – False.
(The Converse of $p \rightarrow q$: $q \rightarrow p$) – If $1+1=1$, then $0+0=0$. – True
(The Inverse of $p \rightarrow q$: $\neg p \rightarrow \neg q$) – If $0+0 \neq 0$, then $1+1 \neq 1$. – True
(The Contrapositive of $p \rightarrow q$: $\neg q \rightarrow \neg p$) – If $1+1 \neq 1$, then $0+0 \neq 0$. – False

(b) If $-1 < 3$ and $3+7=10$, then $\sin(\frac{3\pi}{2}) = -1$. (TRUE)
Converse: If $\sin(\frac{3\pi}{2}) = -1$, then $-1 < 3$ and $3+7=10$. (TRUE)
Inverse: If $-1 \geq 3$ or $3+7 \neq 10$, then $\sin(\frac{3\pi}{2}) \neq -1$. (TRUE)
Contrapositive: If $\sin(\frac{3\pi}{2}) \neq -1$, then $-1 \geq 3$ or $3+7 \neq 10$.

10. (a) True (b) True (c) True

11. a) $(q \rightarrow r) \vee \neg p$ b) $(\neg q \vee r) \vee \neg p$

12.

p	q	$p \veebar q$	$p \wedge \neg q$	$\neg p \wedge q$	$(p \wedge \neg q) \vee (\neg p \wedge q)$	$\neg(p \leftrightarrow q)$
0	0	0	0	0	0	0
0	1	1	0	1	1	1
1	0	1	1	0	1	1
1	1	0	0	0	0	0

13.

p	q	r	$[(p \leftrightarrow q) \wedge (q \leftrightarrow r) \wedge (r \leftrightarrow p)]$	$[(p \rightarrow q) \wedge (q \rightarrow r) \wedge (r \rightarrow p)]$
0	0	0	1	1
0	0	1	0	0
0	1	0	0	0
0	1	1	0	0
1	0	0	0	0
1	0	1	0	0
1	1	0	0	0
1	1	1	1	1

14.

(a)

p	q	$p \wedge q$	$q \rightarrow (p \wedge q)$	$p \rightarrow [q \rightarrow (p \wedge q)]$
0	0	0	1	1
0	1	0	0	1
1	0	0	1	1
1	1	1	1	1

(b) Replace each occurrence of p by $p \vee q$. Then we have the tautology $(p \vee q) \rightarrow [q \rightarrow [(p \vee q) \wedge q]]$ by the first substitution rule. Since $(p \vee q) \wedge q \iff q$, by the absorption laws, it follows that $(p \vee q) \rightarrow [q \rightarrow q] \iff T_0$.

(c)

p	q	$p \vee q$	$p \wedge q$	$q \rightarrow (p \wedge q)$	$(p \vee q) \rightarrow [q \rightarrow (p \wedge q)]$
0	0	0	0	1	1
0	1	1	0	0	0
1	0	1	0	1	1
1	1	1	1	1	1

So the given statement is not a tautology. If we try to apply the second substitution rule to the result in part (a) we would replace the first occurrence of p by $p \vee q$. But this does not result in a tautology because it is not a valid application of this substitution rule – for p is not logically equivalent to $p \vee q$.

15. (a) $\neg p \iff (p \uparrow p)$

(b) $p \vee q \iff \neg(\neg p \wedge \neg q) \iff (\neg p \uparrow \neg q) \iff (p \uparrow p) \uparrow (q \uparrow q)$

(c) $p \wedge q \iff \neg\neg(p \wedge q) \iff \neg(p \uparrow q) \iff (p \uparrow q) \uparrow (p \uparrow q)$

(d) $p \rightarrow q \iff \neg p \vee q \iff \neg(p \wedge \neg q) \iff (p \uparrow \neg q) \iff p \uparrow (q \uparrow q)$

(e) $p \leftrightarrow q \iff (p \rightarrow q) \wedge (q \rightarrow p) \iff t \wedge u \iff (t \uparrow u) \uparrow (t \uparrow u)$, where t stands for $p \uparrow (q \uparrow q)$ and u for $q \uparrow (p \uparrow p)$.

16. (a) $\neg p \iff (p \downarrow p)$

(b) $p \vee q \iff \neg\neg(p \vee q) \iff \neg(p \downarrow q) \iff (p \downarrow q) \downarrow (p \downarrow q)$

(c) $p \wedge q \iff \neg\neg p \wedge \neg\neg q \iff (\neg p \downarrow \neg q) \iff (p \downarrow p) \downarrow (q \downarrow q)$

(d) $p \rightarrow q \iff \neg p \vee q \iff (\neg p \downarrow q) \downarrow (\neg p \downarrow q) \iff [(p \downarrow p) \downarrow q] \downarrow [(p \downarrow p) \downarrow q]$

(e) $p \leftrightarrow q \iff (r \downarrow r) \downarrow (s \downarrow s)$ where r stands for $[(p \downarrow p) \downarrow q] \downarrow [(p \downarrow p) \downarrow q]$ and s for $[(q \downarrow q) \downarrow p] \downarrow [(q \downarrow q) \downarrow p]$

17.

p	q	$\neg(p \downarrow q)$	$(\neg p \uparrow \neg q)$	$\neg(p \uparrow q)$	$(\neg p \downarrow \neg q)$
0	0	0	0	0	0
0	1	1	1	0	0
1	0	1	1	0	0
1	1	1	1	1	1

18.

		Reasons
(a)	$[(p \vee q) \wedge (p \vee \neg q)] \vee q$	
$\Leftrightarrow$	$[p \vee (q \wedge \neg q)] \vee q$	Distributive Law of $\vee$ over $\wedge$
$\Leftrightarrow$	$(p \vee F_0) \vee q$	$q \wedge \neg q \Leftrightarrow F_0$ (Inverse Law)
$\Leftrightarrow$	$p \vee q$	$p \vee F_0 \Leftrightarrow p$ (Identity Law)

		Reasons
(b)	$(p \rightarrow q) \wedge [\neg q \wedge (r \vee \neg q)]$	
$\Leftrightarrow$	$(p \rightarrow q) \wedge \neg q$	Absorption Law (and the Commutative Law of $\vee$)
$\Leftrightarrow$	$(\neg p \vee q) \wedge \neg q$	$p \rightarrow q \Leftrightarrow \neg p \vee q$
$\Leftrightarrow$	$\neg q \wedge (\neg p \vee q)$	Commutative Law of $\wedge$
$\Leftrightarrow$	$(\neg q \wedge \neg p) \vee (\neg q \wedge q)$	Distributive Law of $\wedge$ over $\vee$
$\Leftrightarrow$	$(\neg q \wedge \neg p) \vee F_0$	Inverse Law
$\Leftrightarrow$	$\neg q \wedge \neg p$	Identity Law
$\Leftrightarrow$	$\neg(q \vee p)$	DeMorgan's Laws

19.

		Reasons
(a)	$p \vee [p \wedge (p \vee q)]$	
$\Leftrightarrow$	$p \vee p$	Absorption Law
$\Leftrightarrow$	p	Idempotent Law of $\vee$

		Reasons
(b)	$p \vee q \vee (\neg p \wedge \neg q \wedge r)$	
$\Leftrightarrow$	$(p \vee q) \vee [\neg(p \vee q) \wedge r]$	DeMorgan's Laws
$\Leftrightarrow$	$[(p \vee q) \vee \neg(p \vee q)] \wedge (p \vee q \vee r)$	Distributive Law of $\vee$ over $\wedge$
$\Leftrightarrow$	$T_0 \wedge (p \vee q \vee r)$	Inverse Law
$\Leftrightarrow$	$p \vee q \vee r$	Identity Law

		Reasons
(c)	$[(\neg p \vee \neg q) \rightarrow (p \wedge q \wedge r)]$	
$\Leftrightarrow$	$\neg(\neg p \vee \neg q) \vee (p \wedge q \wedge r)$	$s \rightarrow t \Leftrightarrow \neg s \vee t$
$\Leftrightarrow$	$(\neg\neg p \wedge \neg\neg q) \vee (p \wedge q \wedge r)$	DeMorgan's Laws
$\Leftrightarrow$	$(p \wedge q) \vee (p \wedge q \wedge r)$	Law of Double Negation
$\Leftrightarrow$	$p \wedge q$	Absorption Law

20. (a) $[p \wedge (\neg r \vee q \vee \neg q)] \vee [(r \vee t \vee \neg r) \wedge \neg q] \Longleftrightarrow [p \wedge (\neg r \vee T_0)] \vee [(T_0 \vee t) \wedge \neg q] \Longleftrightarrow (p \wedge T_0) \vee (T_0 \wedge \neg q) \Longleftrightarrow p \vee \neg q$

(b) $[p \vee (p \wedge q) \vee (p \wedge q \wedge r)] \wedge [(p \wedge r \wedge t) \vee t] \Longleftrightarrow p \wedge t$ by the Absorption Law.

Section 2.3

1. (a)

p	q	r	$p \to q$	$(p \lor q)$	$(p \lor q) \to r$
0	0	0	1	0	1
0	0	1	1	0	1
0	1	0	1	1	0
0	1	1	1	1	1
1	0	0	0	1	0
1	0	1	0	1	1
1	1	0	1	1	0
1	1	1	1	1	1

The validity of the argument follows from the results in the last row. (The first seven rows may be ignored.)

(b)

p	q	r	$(p \land q) \to r$	$\neg q$	$p \to \neg r$	$\neg p \lor \neg q$
0	0	0	1	1	1	1
0	0	1	1	1	1	1
0	1	0	1	0	1	1
0	1	1	1	0	1	1
1	0	0	1	1	1	1
1	0	1	1	1	0	1
1	1	0	0	0	1	0
1	1	1	1	0	0	0

The validity of the argument follows from the results in rows 1, 2, and 5 of the table. The results in the other five rows may be ignored.

(c)

p	q	r	$q \lor r$	$p \lor (q \lor r)$	$[p \lor (q \lor r)] \land \neg q$	$p \lor r$
0	0	0	0	0	0	0
0	0	1	1	1	1	1
0	1	0	1	1	0	0
0	1	1	1	1	0	1
1	0	0	0	1	1	1
1	0	1	1	1	1	1
1	1	0	1	1	0	1
1	1	1	1	1	0	1

Consider the last two columns of this truth table. Here we find that whenever the truth value of $[p \lor (q \lor r)] \land \neg q$ is 1 then the truth value of $p \lor r$ is also 1. Consequently,

$$[[p \lor (q \lor r)] \land \neg q] \Rightarrow p \lor r.$$

(The rows of the table that are crucial for assessing the validity of the argument are rows 2, 5, and 6. Rows 1, 3, 4, 7, and 8 may be ignored.)

2. (a)

p	q	r	$p \to q$	$q \to r$	$p \to r$	$[(p \to q) \wedge (q \to r)] \to (p \to r)$
0	0	0	1	1	1	1
0	0	1	1	1	1	1
0	1	0	1	0	1	1
0	1	1	1	1	1	1
1	0	0	0	1	0	1
1	0	1	0	1	1	1
1	1	0	1	0	0	1
1	1	1	1	1	1	1

(b)

p	q	$p \to q$	$(p \to q) \wedge \neg q$	$[(p \to q) \wedge \neg q] \to \neg p$
0	0	1	1	1
0	1	1	0	1
1	0	0	0	1
1	1	1	0	1

(c)

p	q	$\neg p$	$p \vee q$	$(p \vee q) \wedge \neg p$	$[(p \vee q) \wedge \neg p] \to q$
0	0	1	0	0	1
0	1	1	1	1	1
1	0	0	1	0	1
1	1	0	1	0	1

(d)

p	q	r	$p \to r$	$q \to r$	$\overbrace{(p \vee q) \to r}^{s}$	$[(p \to r) \wedge (q \to r)] \to s$
0	0	0	1	1	1	1
0	0	1	1	1	1	1
0	1	0	1	0	0	1
0	1	1	1	1	1	1
1	0	0	0	1	0	1
1	0	1	1	1	1	1
1	1	0	0	0	0	1
1	1	1	1	1	1	1

3. (a) If p has the truth value 0, then so does $p \wedge q$.
(b) When $p \vee q$ has the truth value 0, then the truth value of p (and that of q) is 0.
(c) If q has truth value 0, then the truth value of $[(p \vee q) \wedge \neg p]$ is 0, regardless of the truth value of p.
(d) The statement $q \vee s$ has truth value 0 only when each of q, s has truth value 0. Then

$(p \rightarrow q)$ has truth value 1 when p has truth value 0; $(r \rightarrow s)$ has truth value 1 when r has truth value 0. But then $(p \vee r)$ must have truth value 0, not 1.
(e) For $(\neg p \vee \neg r)$ the truth value is 0 when both p, r have truth value 1. This then forces q, s to have truth value 1, in order for $(p \rightarrow q)$, $(r \rightarrow s)$ to have truth value 1. However, this results in truth value 0 for $(\neg q \vee \neg s)$.

4. (a) Janice's daughter Angela will check Janice's spark plugs. (Modus Ponens)
(b) Brady did not solve the first problem correctly. (Modus Tollens)
(c) This is a **repeat-until** loop. (Modus Ponens)
(d) Tim watched television in the evening. (Modus Tollens)

5. (a) Rule of Conjunctive Simplification
(b) Invalid – attempt to argue by the converse
(c) Modus Tollens
(d) Rule of Disjunctive Syllogism
(e) Invalid – attempt to argue by the inverse

6. (a)

	Steps	Reasons
(1)	$q \wedge r$	Premise
(2)	q	Step (1) and the Rule of Conjunctive Simplification
(3)	$\therefore q \vee r$	Step (2) and the Rule of Disjunctive Amplification

Consequently, $(q \wedge r) \rightarrow (q \vee r)$ is a tautology, or $q \wedge r \Rightarrow q \vee r$.

(b) Consider the truth value assignments $p : 0$, $q : 1$, and $r : 0$. For these assignments $[p \wedge (q \wedge r)] \vee \neg[p \vee (q \wedge r)]$ has truth value 1, while $[p \wedge (q \vee r)] \vee \neg[p \vee (q \vee r)]$ has truth value 0. Therefore, $P \rightarrow P_1$ is *not* a tautology, or $P \not\Rightarrow P_1$.

7.

(1) & (2)	Premise
(3)	Steps (1), (2) and the Rule of Detachment
(4)	Premise
(5)	Step (4) and $(r \rightarrow \neg q) \Longleftrightarrow (\neg\neg q \rightarrow \neg r) \Longleftrightarrow (q \rightarrow \neg r)$
(6)	Steps (3), (5) and the Rule of Detachment
(7)	Premise
(8)	Steps (6), (7) and the Rule of Disjunctive Syllogism
(9)	Step (8) and the Rule of Disjunctive Amplification

8.

(1)	Premise
(2)	Step (1) and the Rule of Conjunctive Simplification
(3)	Premise
(4)	Steps (2), (3) and the Rule of Detachment

(5) Step (1) and the Rule of Conjunctive Simplification
(6) Steps (4), (5) and the Rule of Conjunction
(7) Premise
(8) Step (7) and $[r \rightarrow (s \vee t)] \Longleftrightarrow [\neg(s \vee t) \rightarrow \neg r]$
(9) Step (8) and DeMorgan's Laws
(10) Steps (6), (9) and the Rule of Detachment
(11) Premise
(12) Step (11) and $[(\neg p \vee q) \rightarrow r] \Longleftrightarrow [\neg r \rightarrow \neg(\neg p \vee q)]$
(13) Step (12) and DeMorgan's Laws and the Law of Double Negation
(14) Steps (10), (13) and the Rule of Detachment
(15) Step (14) and the Rule of Conjunctive Simplification

9. (a)

(1) Premise (The Negation of the Conclusion)
(2) Step (1) and $\neg(\neg q \rightarrow s) \Longleftrightarrow \neg(\neg\neg q \vee s) \Longleftrightarrow \neg(q \vee s) \Longleftrightarrow \neg q \wedge \neg s$
(3) Step (2) and the Rule of Conjunctive Simplification
(4) Premise
(5) Steps (3), (4) and the Rule of Disjunctive Syllogism
(6) Premise
(7) Step (2) and the Rule of Conjunctive Simplification
(8) Steps (6), (7) and Modus Tollens
(9) Premise
(10) Steps (8), (9) and the Rule of Disjunctive Syllogism
(11) Steps (5), (10) and the Rule of Conjunction
(12) Step (11) and the Method of Proof by Contradiction

(b)

(1)	$p \rightarrow q$	Premise
(2)	$\neg q \rightarrow \neg p$	Step (1) and $(p \rightarrow q) \Longleftrightarrow (\neg q \rightarrow \neg p)$
(3)	$p \vee r$	Premise
(4)	$\neg p \rightarrow r$	Step (3) and $(p \vee r) \Longleftrightarrow (\neg p \rightarrow r)$
(5)	$\neg q \rightarrow r$	Steps (2), (4) and the Law of the Syllogism
(6)	$\neg r \vee s$	Premise
(7)	$r \rightarrow s$	Step (6) and $(\neg r \vee s) \Longleftrightarrow (r \rightarrow s)$
(8)	$\therefore \neg q \rightarrow s$	Steps (5), (7) and the Law of the Syllogism

(c)

(1)	$\neg p \leftrightarrow q$	Premise
(2)	$(\neg p \rightarrow q) \wedge (q \rightarrow \neg p)$	Step (1) and $(\neg p \leftrightarrow q) \Longleftrightarrow [(\neg p \rightarrow q) \wedge (q \rightarrow \neg p)]$

(3)	$\neg p \to q$	Step (2) and the Rule of Conjunctive Simplification
(4)	$q \to r$	Premise
(5)	$\neg p \to r$	Steps (3), (4) and the Law of the Syllogism
(6)	$\neg r$	Premise
(7)	$\therefore p$	Steps (5), (6) and Modus Tollens.

10. (a)

(1)	$p \wedge \neg q$	Premise
(2)	p	Step (1) and the Rule of Conjunctive Simplification
(3)	r	Premise
(4)	$p \wedge r$	Steps (2), (3) and the Rule of Conjunction
(5)	$\therefore (p \wedge r) \vee q$	Step (4) and the Rule of Disjunctive Amplification

(b)

(1)	$p,\ p \to q$	Premises
(2)	q	Step (1) and the Rule of Detachment
(3)	$\neg q \vee r$	Premise
(4)	$q \to r$	Step (3) and $\neg q \vee r \Longleftrightarrow (q \to r)$
(5)	$\therefore r$	Steps (2), (4) and the Rule of Detachment

(c)

(1)	$p \to q,\ \neg q$	Premises
(2)	$\neg p$	Step (1) and Modus Tollens
(3)	$\neg r$	Premise
(4)	$\neg p \wedge \neg r$	Steps (2), (3) and the Rule of Conjunction
(5)	$\therefore \neg(p \vee r)$	Step (4) and DeMorgan's Laws

(d)

(1)	$r,\ r \to \neg q$	Premises
(2)	$\neg q$	Step (1) and the Rule of Detachment
(3)	$p \to q$	Premise
(4)	$\therefore \neg p$	Steps (2), (3) and Modus Tollens

(e)

(1)	p	Premise
(2)	$\neg q \rightarrow \neg p$	Premise
(3)	$p \rightarrow q$	Step (2) and $(p \rightarrow q) \Longleftrightarrow (\neg q \rightarrow \neg p)$
(4)	q	Steps (1), (3) and the Rule of Detachment
(5)	$p \wedge q$	Steps (1), (4) and the Rule of Conjunction
(6)	$p \rightarrow (q \rightarrow r)$	Premise
(7)	$(p \wedge q) \rightarrow r$	Step (6), and $[p \rightarrow (q \rightarrow r)] \Longleftrightarrow [(p \wedge q) \rightarrow r]$
(8)	$\therefore r$	Steps (5), (7) and the Rule of Detachment

(f)

(1)	$p \wedge q$	Premise
(2)	p	Step (1) and the Rule of Conjunctive Simplification
(3)	$p \rightarrow (r \wedge q)$	Premise
(4)	$r \wedge q$	Steps (2), (3) and the Rule of Detachment
(5)	r	Step (4) and the Rule of Conjunctive Simplification
(6)	$r \rightarrow (s \vee t)$	Premise
(7)	$s \vee t$	Steps (5), (6) and the Rule of Detachment
(8)	$\neg s$	Premise
(9)	$\therefore t$	Steps (7), (8) and the Rule of Disjunctive Syllogism

(g)

(1)	$\neg s,\ p \vee s$	Premises
(2)	p	Step (1) and the Rule of Disjunctive Syllogism
(3)	$p \rightarrow (q \rightarrow r)$	Premise
(4)	$q \rightarrow r$	Steps (2), (3) and the Rule of Detachment
(5)	$t \rightarrow q$	Premise
(6)	$t \rightarrow r$	Steps (4), (5) and the Law of the Syllogism
(7)	$\therefore \neg r \rightarrow \neg t$	Step (6) and $(t \rightarrow r) \Longleftrightarrow (\neg r \rightarrow \neg t)$

(h)

(1)	$\neg p \vee r$	Premise
(2)	$p \rightarrow r$	Step (1) and $(p \rightarrow r) \Leftrightarrow (\neg p \vee r)$
(3)	$\neg r$	Premise
(4)	$\neg p$	Steps (2), (3) and Modus Tollens
(5)	$p \vee q$	Premise
(6)	$\neg p \rightarrow q$	Step (5) and $(p \vee q) \Leftrightarrow (\neg\neg p \vee q) \Leftrightarrow (\neg p \rightarrow q)$
(7)	$\therefore q$	Steps (4), (6) and Modus Ponens

11. (a) $p:1 \quad q:0 \quad r:1$
(b) $p:0 \quad q:0 \quad r:0$ or 1
$p:0 \quad q:1 \quad r:1$
(c) $p,q,r:1 \quad s:0$
(d) $p,q,r:1 \quad s:0$

12. a) p: Rochelle gets the supervisor's position.
q: Rochelle works hard.
r: Rochelle gets a raise.
s: Rochelle buys a new car.

$$\begin{array}{l} (p \wedge q) \rightarrow r \\ r \rightarrow s \\ \underline{\neg s} \\ \therefore \neg p \vee \neg q \end{array}$$

(1)	$\neg s$	Premise
(2)	$r \rightarrow s$	Premise
(3)	$\neg r$	Steps (1), (2) and Modus Tollens
(4)	$(p \wedge q) \rightarrow r$	Premise
(5)	$\neg(p \wedge q)$	Steps (3), (4) and Modus Tollens
(6)	$\therefore \neg p \vee \neg q$	Step (5) and $\neg(p \wedge q) \Longleftrightarrow \neg p \vee \neg q$.

b) p: Dominic goes to the racetrack.
q: Helen gets mad.
r: Ralph plays cards all night.
s: Carmela gets mad.
t: Veronica is notified.

$$\begin{array}{l} p \rightarrow q \\ r \rightarrow s \\ (q \vee s) \rightarrow t \\ \underline{\neg t} \\ \therefore \neg p \wedge \neg r \end{array}$$

(1)	$\neg t$	Premise
(2)	$(q \vee s) \rightarrow t$	Premise
(3)	$\neg(q \vee s)$	Steps (1), (2) and Modus Tollens
(4)	$\neg q \wedge \neg s$	Step (3) and $\neg(q \vee s) \Longleftrightarrow \neg q \wedge \neg s$
(5)	$\neg q$	Step (4) and the Rule of Conjunctive Simplification
(6)	$p \rightarrow q$	Premise
(7)	$\neg p$	Steps (5), (6) and Modus Tollens
(8)	$\neg s$	Step (4) and the Rule of Conjunctive Simplification

(9)	$r \rightarrow s$	Premise
(10)	$\neg r$	Steps (8), (9) and Modus Tollens
(11)	$\therefore \neg p \wedge \neg r$	Steps (7), (10) and the Rule of Conjunction

c) p: There is a chance of rain.
q: Lois' red head scarf is missing.
r: Lois does not mow her lawn.
s: The temperature is over 80° F.

$$\begin{array}{l} (p \vee q) \rightarrow r \\ s \rightarrow \neg p \\ s \wedge \neg q \\ \hline \therefore \neg r \end{array}$$

The following truth value assignments provide a counterexample to the validity of this argument:
$p: 0$; $q: 0$; $r: 1$; $s: 1$

13. (a)

p	q	r	$p \vee q$	$\neg p \vee r$	t: $(p \vee q) \wedge (\neg p \vee r)$	$q \vee r$	$t \rightarrow (q \vee r)$
0	0	0	0	1	0	0	1
0	0	1	0	1	0	1	1
0	1	0	1	1	1	1	1
0	1	1	1	1	1	1	1
1	0	0	1	0	0	0	1
1	0	1	1	1	1	1	1
1	1	0	1	0	0	1	1
1	1	1	1	1	1	1	1

From the last column of the truth table it follows that $[(p \vee q) \wedge (\neg p \vee r)] \rightarrow (q \vee r)$ is a tautology.

Alternately we can try to see if there are truth values that can be assigned to p, q, and r so that $(q \vee r)$ has truth value 0 while $(p \vee q)$, $(\neg p \vee r)$ both have truth value 1.

For $(q \vee r)$ to have truth value 0, it follows that $q: 0$ and $r: 0$. Consequently, for $(p \vee q)$ to have truth value 1, we have $p: 1$ since $q: 0$. Likewise, with $r: 0$ it follows that $\neg p: 1$ if $(\neg p \vee) r$ has truth value 1. But we cannot have $p: 1$ and $\neg p: 1$. So whenever $(p \vee q)$, $(\neg p \vee r)$ have truth value 1, we have $(q \vee r)$ with truth value 1 and it follows that $[(p \vee q) \wedge (\neg p \vee r)] \rightarrow (q \vee r)$ is a tautology.

Finally we can also argue as follows:

Steps		Reasons
1.	$p \vee q$	1. Premise
2.	$q \vee p$	2. Step (1) and the Commutative Law of $\vee$
3.	$\neg(\neg q) \vee p$	3. Step (2) and the Law of Double Negation
4.	$\neg q \rightarrow p$	4. Step (3), $\neg q \rightarrow p \Leftrightarrow \neg(\neg q) \vee p$
5.	$\neg p \vee r$	5. Premise
6.	$p \rightarrow r$	6. Step (5), $p \rightarrow r \Leftrightarrow \neg p \vee r$
7.	$\neg q \rightarrow r$	7. Steps (4), (6), and the Law of the Syllogism
8.	$\therefore q \vee r$	8. Step (7), $\neg q \rightarrow r \Leftrightarrow q \vee r$

(b)

(i)

Steps		Reasons
1.	$p \vee (q \vee r)$	1. Premise
2.	$(p \vee q) \wedge (p \vee r)$	2. Step (1) and the Distribution Law of $\vee$ over $\wedge$
3.	$p \vee r$	3. Step (2) and the Rule of Conjunctive Simplification
4.	$p \rightarrow s$	4. Premise
5.	$\neg p \vee s$	5. Step (4), $p \rightarrow s \Leftrightarrow \neg p \vee s$
6.	$\therefore r \vee s$	6. Steps (3), (5), the Rule of Conjunction, and Resolution

(ii)

Steps		Reasons
1.	$p \leftrightarrow q$	1. Premise
2.	$(p \rightarrow q) \wedge (q \rightarrow p)$	2. $(p \leftrightarrow q) \Leftrightarrow [(p \rightarrow q) \wedge (q \rightarrow p)]$
3.	$p \rightarrow q$	3. Step (2) and the rule of Conjunctive Simplification
4.	$\neg p \vee q$	4. Step (3), $p \rightarrow q \Leftrightarrow \neg p \vee q$
5.	p	5. Premise
6.	$p \vee q$	7. Step (5) and the Rule of Disjunctive Amplification
7.	$[(p \vee q) \wedge (\neg p \vee q)]$	7. Steps (6), (4), and the Rule of Conjunction
8.	$q \vee q$	8. Step (7) and Resolution
9.	$\therefore q$	9. Step (8) and the Idempotent Law of $\vee$.

(iii)

Steps		Reasons
1.	$p \vee q$	1. Premise
2.	$p \rightarrow r$	2. Premise
3.	$\neg p \vee r$	3. Step (2), $p \rightarrow r \Leftrightarrow \neg p \vee r$
4.	$[(p \vee q) \wedge (\neg p \vee r)]$	4. Steps (1), (3), and the Rule of Conjunction
5.	$q \vee r$	5. Step (4) and Resolution
6.	$r \rightarrow s$	6. Premise
7.	$\neg r \vee s$	7. Step (6), $r \rightarrow s \Leftrightarrow \neg r \vee s$
8.	$[(r \vee q) \wedge (\neg r \vee s)]$	8. Steps (5), (7), the Commutative Law of $\vee$, and the Rule of Conjunction
9.	$\therefore q \vee s$	9. Step (8) and Resolution

(iv)

	Steps		Reasons
1.	$\neg p \vee q \vee r$	1.	Premise
2.	$q \vee (\neg p \vee r)$	2.	Step (1) and the Commutative and Associative Laws of $\vee$
3.	$\neg q$	3.	Premise
4.	$\neg q \vee (\neg p \vee r)$	4.	Step (3) and the Rule of Disjunctive Amplification
5.	$[[q \vee (\neg \vee r)] \wedge [\neg q \vee (\neg p \vee r)]]$	5.	Steps (2), (4), and the Rule of Conjunction
6.	$(\neg p \vee r)$	6.	Step (5), Resolution, and the Idempotent Law of $\wedge$
7.	$\neg r$	7.	Premise
8.	$\neg r \vee \neg p$	8.	Step (7) and the Rule of Disjunctive Amplification
9.	$[(r \vee \neg p) \wedge (\neg r \vee \neg p)]$	9.	Steps (6), (8), the Commutative Law of $\vee$, and the Rule of Conjunction
10.	$\therefore \neg p$	10.	Step (9), Resolution, and the Idempotent Law of $\vee$

(v)	Steps		Reasons
1.	$\neg p \vee s$	1.	Premise
2.	$p \vee q \vee t$	2.	Premise
3.	$p \vee (q \vee t)$	3.	Step (2) and the Associative Law of $\vee$
4.	$[[p \vee (q \vee t)] \wedge (\neg p \vee s)]$	4.	Steps (3), (1), and the Rule of Conjunction
5.	$(q \vee t) \vee s$	5.	Step (4) and Resolution (and the First Substitution Rule)
6.	$q \vee (t \vee s)$	6.	Step (5) and the Associative Law of $\vee$
7.	$\neg q \vee r$	7.	Premise
8.	$[[q \vee (t \vee s)]$	8.	Steps (6), (7), and the Rule of Conjunction
9.	$(t \vee s) \vee r$	9.	Step (8) and Resolution (and the First Substitution Rule)
10.	$t \vee (s \vee r)$	10.	Step (9) and the Associative Law of $\vee$
11.	$\neg t \vee (s \wedge r)$	11.	Premise
12.	$(\neg t \vee s) \wedge (\neg t \vee r)$	12.	Step (11) and the Distributive Law of $\vee$ over $\wedge$
13.	$\neg t \vee s$	13.	Step (12) and the Rule of Conjunctive Simplification
14.	$[[t \vee (s \vee r)] \wedge (\neg t \vee s)]$	14.	Steps (10), (13), and the Rule of Conjunction
15.	$(s \vee r) \vee s$	15.	Step (14) and Resolution (and the First Substitution Rule)
16.	$\therefore r \vee s$	16.	Step (15) and the Commutative, Associative, and Idempotent Laws of $\vee$

(c) Consider the following assignments.

p: Jonathan has his driver's license.
q: Jonathan's new car is out of gas.
r: Jonathan likes to drive his new car.

Then the given argument can be written in symbolic form as

$$\begin{array}{l} \neg p \vee q \\ p \vee \neg r \\ \underline{\neg q \vee \neg r} \\ \therefore \neg r \end{array}$$

Steps		Reasons
1.	$\neg p \vee q$	1. Premise
2.	$p \vee \neg r$	2. Premise
3.	$(p \vee \neg r) \wedge (\neg p \vee q)$	3. Steps (2), (1), and the Rule of Conjunction
4.	$\neg r \vee q$	4. Step (3) and Resolution
5.	$q \vee \neg r$	5. Step (4) and the Commutative Law of $\vee$
6.	$\neg q \vee \neg r$	6. Premise
7.	$(q \vee \neg r) \wedge (\neg q \vee \neg r)$	7. Steps (5), (6), and the Rule of Conjunction
8.	$\neg r \vee \neg r$	8. Step (7) and Resolution
9.	$\therefore \neg r$	9. Step (8) and Idempotent Law of $\vee$

Section 2.4

1.

(a)	False	(b)	False	(c)	False
(d)	True	(e)	False	(f)	False

2. (a) (i) True (ii) True (iii) True (iv) True

(b) The only substitution for x that makes the open statement $[p(x) \wedge q(x)] \wedge r(x)$ into a true statement is $x = 2$.

3. Statements (a), (c), and (e) are true, while statements (b), (d), and (f) are false.

4. (a) Every polygon is a quadrilateral or a triangle (but not both). (True — for this universe.)
(b) Every isosceles triangle is equilateral. (False)
(c) There exists a triangle with an interior angle that exceeds 180°. (False)
(d) A triangle has all of its interior angles equal if and only if it is an equilateral triangle. (True)
(e) There exists a quadrilateral that is not a rectangle. (True)
(f) There exists a rectangle that is not a square. (True)
(g) If all the sides of a polygon are equal, then the polygon is an equilateral triangle. (False)
(h) No triangle has an interior angle that exceeds 180°. (True)
(i) A polygon (of three or four sides) is a square if and only if all of its interior angles are equal and all of its sides are equal. (False)

(j) A triangle has all interior angles equal if and only if all of its sides are equal. (True)

5.

(a)	$\exists x\ [m(x) \wedge c(x) \wedge j(x)]$	True
(b)	$\exists x\ [s(x) \wedge c(x) \wedge \neg m(x)]$	True
(c)	$\forall x\ [c(x) \rightarrow (m(x) \veebar p(x))]$	False
(d)	$\forall x\ [(g(x) \wedge c(x)) \rightarrow \neg p(x)]$,	True
	or $\forall x\ [(p(x) \wedge c(x)) \rightarrow \neg g(x)]$,	
	or $\forall x\ [(g(x) \wedge p(x)) \rightarrow \neg c(x)]$	
(e)	$\forall x\ [(c(x) \wedge s(x)) \rightarrow (p(x) \veebar e(x))]$,	True

6.

(a) True (b) True (c) False
(d) True (e) False (f) False

7. (a)
- (i) $\exists x\ q(x)$
- (ii) $\exists x\ [p(x) \wedge q(x)]$
- (iii) $\forall x\ [q(x) \rightarrow \neg t(x)]$
- (iv) $\forall x\ [q(x) \rightarrow \neg t(x)]$
- (v) $\exists x\ [q(x) \wedge t(x)]$
- (vi) $\forall x\ [(q(x) \wedge r(x)) \rightarrow s(x)]$

(b) Statements (i), (iv), (v), and (vi) are true. Statements (ii) and (iii) are false: $x = 10$ provides a counterexample for either statement.

(c)
- (i) If x is a perfect square, then $x > 0$.
- (ii) If x is divisible by 4, then x is even.
- (iii) If x is divisible by 4, then x is not divisible by 5.
- (iv) There exists an integer that is divisible by 4 but it is not a perfect square.

(d) (i) Let $x = 0$. (iii) Let $x = 20$.

8. (a) True (b) False: For $x = 1$, $q(x)$ is true while $p(x)$ is false.
(c) True (d) True (e) True (f) True
(g) True (h) False: For $x = -1$, $(p(x) \vee q(x))$ is true but $r(x)$ is false.

9.

(a)	(i) True	(ii)	False –	Consider $x = 3$.
	(iii) True	(iv)	True	
(b)	(i) True	(ii)	False –	Consider $x = 3$.
	(iii) True	(iv)	True	
(c)	(i) True	(ii)	True	
	(iii) True	(iv)	False –	For $x = 2$ or 5, the truth value of $p(x)$ is 1 while that of $r(x)$ is 0.

10. (a) $\forall m, n\ \ A[m, n] > 0$
(b) $\forall m, n\ \ 0 < A[m, n] \leq 70$

(c) $\exists m, n \; A[m,n] > 60$
(d) $\forall m \; [(1 \leq n < 19) \rightarrow (A[m,n] < A[m,n+1])]$
(e) $\forall n \; [(1 \leq m < 9) \rightarrow (A[m,n] < A[m+1,n])]$
(f) $\forall \; 1 \leq m, i \leq 3 \; \forall \; 1 \leq n, j \leq 20 \; [((m,n) \neq (i,j)) \rightarrow (A[m,n] \neq A[i,j])]$

11. (a) In this case the variable x is free while the variables y, z are bound.
(b) Here the variables x, y are bound; the variable z is free.

12. (a)

(i)	False	(ii)	True	(iii)	True
(iv)	False, if $x = 0$	(v)	False, if $x = 0$	(vi)	True

(vii) False — If $y = 0$ then $x \neq 0$; if $y \neq 0$, let $x = 2y$.
(viii) False — Let $x = 2$ and $y = -2$, for example.

(b) Statements (iv), (v), and (viii) are now true — because of the change in universe.

(c) (i) True (ii) True (iii) True
(iv) False — For any y consider $x = 2y$.

13. (a) $p(2,3) \wedge p(3,3) \wedge p(5,3)$
(b) $[p(2,2) \vee p(2,3) \vee p(2,5)] \vee [p(3,2) \vee p(3,3) \vee p(3,5)] \vee [p(5,2) \vee p(5,3) \vee p(5,5)]$
(c) $[p(2,2) \vee p(3,2) \vee p(5,2)] \wedge [p(2,3) \vee p(3,3) \vee p(5,3)] \wedge [p(2,5) \vee p(3,5) \vee p(5,5)]$

14. Statements (a), (b), (e), and (f) are logically equivalent and each may be expressed as $\forall n[q(n) \rightarrow p(n)]$. Statements (c), (g) are logically equivalent and each may be expressed as $\forall n[p(n) \rightarrow q(n)]$. Statement (d) is not logically equivalent to any of the other six statements.

15. a) The proposed negation is correct and is a true statement.
b) The proposed negation is wrong. A correct version of the negation is: For all rational numbers x, y, the sum $x + y$ is rational. This correct version of the negation is a true statement.
c) The proposed negation is correct — but false. The (original) statement is true.
d) The proposed negation is wrong. A correct version of the negation is: For all integers x, y, if x, y are both odd, then xy is even.
The (original) statement is true.

16. (a) Some student in Professor Lenhart's C++ class is not majoring in either computer science or mathematics.
(b) If a student is in Professor Lenhart's C++ class, then that student is not majoring in history.
or, No student majoring in history is in Professor Lenhart's C++ class.

17. a) There exists an integer n such that n is not divisible by 2 but n is even (that is, not odd).
b) There exist integers k, m, n such that $k - m$ and $m - n$ are odd, and $k - n$ is odd.

c) For some real number x, $x^2 > 16$ but $-4 \leq x \leq 4$ (that is, $-4 \leq x$ and $x \leq 4$).
d) There exists a real number x such that $|x-3| < 7$ and either $x \leq -4$ or $x \geq 10$.

18. (a) $\forall x\ [\neg p(x) \wedge \neg q(x)]$
(b) $\exists x\ [\neg p(x) \vee q(x)]$
(c) $\exists x\ [p(x) \wedge \neg q(x)]$
(d) $\forall x\ [(p(x) \vee q(x)) \wedge \neg r(x)]$

19. (a) Statement: For all positive integers m, n, if $m > n$ then $m^2 > n^2$. (TRUE)
Converse: For all positive integers m, n, if $m^2 > n^2$ then $m > n$. (TRUE)
Inverse: For all positive integers m, n, if $m \leq n$ then $m^2 \leq n^2$. (TRUE)
Contrapositive: For all positive integers m, n, if $m^2 \leq n^2$ then $m \leq n$. (TRUE)
(b) Statement: For all integers a, b, if $a > b$ then $a^2 > b^2$. (FALSE — let $a = 1$ and $b = -2$.)
Converse: For all integers a, b, if $a^2 > b^2$ then $a > b$. (FALSE — let $a = -5$ and $b = 3$.)
Inverse: For all integers a, b, if $a \leq b$ then $a^2 \leq b^2$. (FALSE — let $a = -5$ and $b = 3$.)
Contrapositive: For all integers a, b, if $a^2 \leq b^2$ then $a \leq b$. (FALSE — let $a = 1$ and $b = -2$.)
(c) Statement: For all integers m, n, and p, if m divides n and n divides p then m divides p. (TRUE)
Converse: For all integers m and p, if m divides p, then for each integer n it follows that m divides n and n divides p. (FALSE — let $m = 1$, $n = 2$, and $p = 3$.)
Inverse: For all integers m, n, and p, if m does not divide n or n does not divide p, then m does not divide p. (False — let $m = 1$, $n = 2$, and $p = 3$.)
Contrapositive: For all integers m and p, if m does not divide p, then for each integer n it follows that m does not divide n or n does not divide p. (TRUE)
(d) Statement: $\forall x\ [(x > 3) \rightarrow (x^2 > 9)]$ (TRUE)
Converse: $\forall x\ [(x^2 > 9) \rightarrow (x > 3)]$ (FALSE — let $x = -5$.)
Inverse: $\forall x\ [(x \leq 3) \rightarrow (x^2 \leq 9)]$ (FALSE — let $x = -5$.)
Contrapositive: $\forall x\ [(x^2 \leq 9) \rightarrow (x \leq 3)]$ (TRUE)
(e) Statement: $\forall x\ [(x^2 + 4x - 21 > 0) \rightarrow [(x > 3) \vee (x < -7)]]$ (TRUE)
Converse: $\forall x\ [[(x > 3) \vee (x < -7)] \rightarrow (x^2 + 4x - 21 > 0)]$ (TRUE)
Inverse: $\forall x\ [(x^2 + 4x - 21 \leq 0) \rightarrow [(x \leq 3) \wedge (x \geq -7)]]$, or $\forall x\ [(x^2 + 4x - 21 \leq 0) \rightarrow (-7 \leq x \leq 3)]$ (TRUE)
Contrapositive: $\forall x\ [[(x \leq 3) \wedge (x \geq -7)] \rightarrow (x^2 + 4x - 21 \leq 0)]$, or $\forall x\ [(-7 \leq x \leq 3) \rightarrow (x^2 + 4x - 21 \leq 0)]$ (TRUE)

20. For each of the following answers it is possible to have the implication and its contrapositive interchanged. When this happens the corresponding converse and inverse must also be interchanged.

(a) Implication: If a positive integer is divisible by 21, then it is divisible by 7. (TRUE)
Converse: If a positive integer is divisible by 7, then it is divisible by 21. (FALSE — consider the positive integer 14.)

Inverse: If a positive integer is not divisible by 21, then it is not divisible by 7. (FALSE — consider the positive integer 14.)
Contrapositive: If a positive integer is not divisible by 7, then it is not divisible by 21. (TRUE)
(b) Implication: If a snake is a cobra, then it is dangerous.
Converse: If a snake is dangerous, then it is a cobra.
Inverse: If a snake is not a cobra, then it is not dangerous.
Contrapositive: If a snake is not dangerous, then it is not a cobra.
(c) Implication: For each complex number z, if z^2 is real then z is real. (FALSE — let $z = i$.)
Converse: For each complex number z, if z is real then z^2 is real. (TRUE)
Inverse: For each complex number z, if z^2 is not real then z is not real. (TRUE)
Contrapositive: For each complex number z, if z is not real then z^2 is not real. (FALSE — let $z = i$.)

21. (a) True (b) False (c) False (d) True (e) False

22. (a) True (b) False (c) True (d) True (e) True

23. (a) $\forall a\, \exists b\, [a + b = b + a = 0]$
(b) $\exists u\, \forall a\, [au = ua = a]$
(c) $\forall a \neq 0\, \exists b\, [ab = ba = 1]$
(d) The statement in part (b) remains true but the statement in part (c) is no longer true for this new universe.

24. (a) True (b) False (c) False (d) True

25. (a) $\exists x\, \exists y\, [(x > y) \wedge (x - y \leq 0)]$
(b) $\exists x\, \exists y\, [(x < y) \wedge \forall z [x \geq z \vee z \geq y]]$
(c) $\exists x\, \exists y\, [(|x| = |y|) \wedge (y \neq \pm x)]$

26. $\lim_{n\to\infty} r_n \neq L \Leftrightarrow \exists \epsilon > 0\, \forall k > 0\, \exists n\, [(n > k) \wedge |r_n - L| \geq \epsilon]$

Section 2.5

1. Although we may write $28 = 25 + 1 + 1 + 1 = 16 + 4 + 4 + 4$, there is no way to express 28 as the sum of at most three perfect squares.

2. Although $3 = 1 + 1 + 1$ and $5 = 4 + 1$, when we get to 7 there is a problem. We can write $7 = 4 + 1 + 1 + 1$, but we cannot write 7 as the sum of three or fewer perfect squares. [There is also a problem with the integers 15 and 23.]

3. Here we find that

$30 = 25 + 4 + 1$	$40 = 36 + 4$	$50 = 25 + 25$
$32 = 16 + 16$	$42 = 25 + 16 + 1$	$52 = 36 + 16$
$34 = 25 + 9$	$44 = 36 + 4 + 4$	$54 = 25 + 25 + 4$
$36 = 36$	$46 = 36 + 9 + 1$	$56 = 36 + 16 + 4$
$38 = 36 + 1 + 1$	$48 = 16 + 16 + 16$	$58 = 49 + 9$

4.

$4 = 2 + 2$	$16 = 13 + 3$	$28 = 23 + 5$
$6 = 3 + 3$	$18 = 13 + 5$	$30 = 17 + 13$
$8 = 3 + 5$	$20 = 17 + 3$	$32 = 19 + 13$
$10 = 5 + 5$	$22 = 17 + 5$	$34 = 17 + 17$
$12 = 7 + 5$	$24 = 17 + 7$	$36 = 19 + 17$
$14 = 7 + 7$	$26 = 19 + 7$	$38 = 19 + 19$

5. (a) The real number π is not an integer.
(b) Margaret is a librarian.
(c) All administrative directors know how to delegate authority.
(d) Quadrilateral $MNPQ$ is not equiangular.

6. (a) Valid — This argument follows from the Rule of Universal Specification and Modus Ponens.
(b) Invalid — Attempt to argue by the converse.
(c) Invalid — Attempt to argue by the inverse.

7. (a) When the statement $\exists x\ [p(x) \vee q(x)]$ is true, there is at least one element c in the prescribed universe where $p(c) \vee q(c)$ is true. Hence at least one of the statements $p(c), q(c)$ has the truth value 1, so at least one of the statements $\exists x\ p(x)$ and $\exists x\ q(x)$ is true. Therefore, it follows that $\exists x\ p(x) \vee \exists x\ q(x)$ is true, and $\exists x\ [p(x) \vee q(x)] \Longrightarrow \exists x\ p(x) \vee \exists x\ q(x)$. Conversely, if $\exists x\ p(x) \vee \exists x\ q(x)$ is true, then at least one of $p(a)$, $q(b)$ has truth value 1, for some a, b in the prescribed universe. Assume without loss of generality that it is $p(a)$. Then $p(a) \vee q(a)$ has truth value 1 so $\exists x\ [p(x) \vee q(x)]$ is a true statement, and $\exists x\ p(x) \vee \exists x\ q(x) \Longrightarrow \exists x\ [p(x) \vee q(x)]$.
(b) First consider when the statement $\forall x\ [p(x) \wedge q(x)]$ is true. This occurs when $p(a) \wedge q(a)$ is true for each a in the prescribed universe. Then $p(a)$ is true (as is $q(a)$) for all a in the universe, so the statements $\forall x\ p(x)$, $\forall x\ q(x)$ are true. Therefore, the statement $\forall x\ p(x) \wedge \forall x\ q(x)$ is true and $\forall x\ [p(x) \wedge q(x)] \Longrightarrow \forall x\ p(x) \wedge \forall x\ q(x)$. Conversely, suppose that $\forall x\ p(x) \wedge \forall x\ q(x)$ is a true statement. Then $\forall x\ p(x)$, $\forall x\ q(x)$ are both true. So now let c be any element in the prescribed universe. Then $p(c)$, $q(c)$, and $p(c) \wedge q(c)$ are all true. And, since c was chosen arbitrarily, it follows that the statement $\forall x\ [p(x) \wedge q(x)]$ is true, and $\forall x\ p(x) \wedge \forall x\ q(x) \Longrightarrow \forall x\ [p(x) \wedge q(x)]$.

8. (a) Suppose that the statement $\forall x\ p(x) \vee \forall x\ q(x)$ is true, and suppose without loss of generality that $\forall x\ p(x)$ is true. Then for each c in the given universe $p(c)$ is true, as is

$p(c) \vee q(c)$. Hence $\forall x\ [p(x) \vee q(x)]$ is true and $\forall x\ p(x) \vee \forall x\ q(x) \Longrightarrow \forall x\ [p(x) \vee q(x)]$.
(b) Let $p(x):\ x > 0$ and $q(x):\ x < 0$ for the universe of all nonzero integers. Then $\forall x\ p(x), \forall x\ q(x)$ are false, so $\forall x\ p(x) \vee \forall x\ q(x)$ is false, while $\forall x\ [p(x) \vee q(x)]$ is true.

9. (1) Premise
(2) Premise
(3) Step (1) and the Rule of Universal Specification
(4) Step (2) and the Rule of Universal Specification
(5) Step (4) and the Rule of Conjunctive Simplification
(6) Steps (5), (3), and Modus Ponens
(7) Step (6) and the Rule of Conjunctive Simplification
(8) Step (4) and the Rule of Conjunctive Simplification
(9) Steps (7), (8), and the Rule of Conjunction
(10) Step (9) and the Rule of Universal Generalization

10.
(4) Step (1) and the Rule of Universal Specification
(5) Steps (3), (4), and the Rule of Disjunctive Syllogism
(6) Premise
(7) Step (6) and the Rule of Universal Specification
(8) Step (7) and $\neg q(a) \vee r(a) \Leftrightarrow q(a) \rightarrow r(a)$
(9) Steps (5), (8), and Modus Ponens (or the Rule of Detachment)
(10) Premise
(11) Step (10) and the Rule of Universal Specification
(12) Step (11) and $s(a) \rightarrow \neg r(a) \Leftrightarrow \neg\neg r(a) \rightarrow \neg s(a) \Leftrightarrow r(a) \rightarrow \neg s(a)$
(13) Steps (9), (12), and Modus Ponens (or the Rule of Detachment)

11. Consider the open statements
$w(x)$: x works for the credit union
$\ell(x)$: x writes loan applications
$c(x)$: x knows COBOL
$q(x)$: x knows Excel
and let r represent Roxe and i represent Imogene.

In symoblic form the given argument is given as follows:

$$\begin{array}{l} \forall x\ [w(x) \rightarrow c(x)] \\ \forall x\ [(w(x) \wedge \ell(x)) \rightarrow q(x)] \\ w(r) \wedge \neg q(r) \\ \underline{q(i) \wedge \neg c(i)} \\ \therefore \neg \ell(r) \wedge \neg w(i) \end{array}$$

The steps (and reasons) needed to verify this argument can now be presented.

	Steps	**Reasons**
(1)	$\forall x\ [w(x) \rightarrow c(x)]$	Premise
(2)	$q(i) \wedge \neg c(i)$	Premise
(3)	$\neg c(i)$	Step (2) and the Rule of Conjunctive Simplification
(4)	$w(i) \rightarrow c(i)$	Step (1) and the Rule of Universal Specification
(5)	$\neg w(i)$	Steps (3), (4), and Modus Tollens
(6)	$\forall x\ [(w(x) \wedge \ell(x)) \rightarrow q(x)]$	Premise
(7)	$w(r) \wedge \neg q(r)$	Premise
(8)	$\neg q(r)$	Step (7) and the Rule of Conjunctive Simplification
(9)	$(w(r) \wedge \ell(r)) \rightarrow q(r)$	Step (6) and the Rule of Universal Specification
(10)	$\neg(w(r) \wedge \ell(r))$	Steps (8), (9), and Modus Tollens
(11)	$w(r)$	Step (7) and the Rule of Conjunctive Simplification
(12)	$\neg w(r) \vee \neg \ell(r)$	Step (10) and DeMorgan's Law
(13)	$\neg \ell(r)$	Steps (11), (12), and the Rule of Disjunctive Syllogism
(14)	$\therefore\ \neg \ell(r) \wedge \neg w(i)$	Steps (13), (5), and the Rule of Conjunction

12. (a) Proof: Since k, ℓ are both even we may write $k = 2c$ and $\ell = 2d$, where c, d are integers. This follows from Definition 2.8. Then the sum $k + \ell = 2c + 2d = 2(c + d)$ by the distributive law of multiplication over addition for integers. Consequently, by Definition 2.8, it follows from $k + \ell = 2(c + d)$, with $c + d$ an integer, that $k + \ell$ is even.
(b) Proof: As in part (a) we write $k = 2c$ and $\ell = 2d$ for integers c, d. Then — by the commutative and associative laws of multiplication for integers — the product $k\ell = (2c)(2d) = 2(2cd)$, where $2cd$ is an integer. With $(2c)(2d) = 2(2cd)$, and $2cd$ an integer, it now follows from Definition 2.8 that $k\ell$ is even.

13. (a) Contrapositive: For all integers k and ℓ, if k, ℓ are not both odd then $k\ell$ is not odd. — OR, For all integers k and ℓ, if at least one of k, ℓ is even then $k\ell$ is even.
Proof: Let us assume (without loss of generality) that k is even. Then $k = 2c$ for some integer c — because of Definition 2.8. Then $k\ell = (2c)\ell = 2(c\ell)$, by the associative law of multiplication for integers — and $c\ell$ is an integer. Consequently, $k\ell$ is even — once again, by Definition 2.8. [Note that this result does not require anything about the integer ℓ.]

(b) Contrapositive: For all integers k and ℓ, if k and ℓ are not both even or both odd then $k + \ell$ is odd. — OR, For all integers k and ℓ, if one of k, ℓ is odd and the other even then $k + \ell$ is odd.
Proof: Let us assume (without loss of generality) that k is even and ℓ is odd. Then it follows from Definition 2.8 that we may write $k = 2c$ and $\ell = 2d + 1$ for integers c and d. And now we find that $k + \ell = 2c + (2d + 1) = 2(c + d) + 1$, where $c + d$ is an integer — by the associative law of addition and the distributive law of multiplication over addition for integers. From Definition 2.8 we find that $k + \ell = 2(c + d) + 1$ implies that $k + \ell$ is odd.

14. Proof: Since n is odd we may write $n = 2a + 1$, where a is an integer — by Definition 2.8. Then $n^2 = (2a + 1)^2 = 4a^2 + 4a + 1 = 2(2a^2 + 2a) + 1$, where $2a^2 + 2a$ is an integer. So again by Definition 2.8 it follows that n^2 is odd.

15. Proof: Assume that for some integer n, n^2 is odd while n is not odd. Then n is even and we may write $n = 2a$, for some integer a — by Definition 2.8. Consequently, $n^2 = (2a)^2 = (2a)(2a) = (2 \cdot 2)(a \cdot a)$, by the commutative and associative laws of multiplication for integers. Hence, we may write $n^2 = 2(2a^2)$, with $2a^2$ an integer — and this means that n^2 is even. Thus we have arrived at a contradiction since we now have n^2 both odd (at the start) and even. This contradiction came about from the false assumption that n is not odd. Therefore, for every integer n, it follows that n^2 odd $\Rightarrow n$ odd.

16. Here we must prove two results — namely, (i) if n^2 is even, then n is even; and (ii) if n is even, then n^2 is even.
Proof (i): Using the method of contraposition, suppose that n is not even — that is, n is odd. Then $n = 2a+1$, for some integer a, and $n^2 = (2a+1)^2 = 4a^2+4a+1 = 2(2a^2+2a)+1$, where $2a^2 + 2a$ is an integer. Hence n^2 is odd (or, not even).
Proof (ii): If n is even then $n = 2c$ for some integer c. So $n^2 = (2c)^2 = (2c)(2c) = 2(c(2c)) = 2((c \cdot 2)c) = 2((2c)c) = 2(2c^2)$, by the associative and commutative laws of multiplication for integers. Since $2c^2$ is an integer, it follows that n^2 is even.

17. Proof:
(1) Since n is odd we have $n = 2a + 1$ for some integer a. Then $n + 11 = (2a + 1) + 11 = 2a + 12 = 2(a + 6)$, where $a + 6$ is an integer. So by Definition 2.8 it follows that $n + 11$ is even.
(2) If $n + 11$ is not even, then it is odd and we have $n + 11 = 2b + 1$, for some integer b. So $n = (2b + 1) - 11 = 2b - 10 = 2(b - 5)$, where $b - 5$ is an integer, and it follows from Definition 2.8 that n is even — that is, not odd.
(3) In this case we stay with the hypothesis — that n is odd — and also assume that $n + 11$ is not even — hence, odd. So we may write $n + 11 = 2b + 1$, for some integer b. This then implies that $n = 2(b - 5)$, for the integer $b - 5$. So by Definition 2.8 it follows that n is even. But with n both even (as shown) and odd (as in the hypothesis) we have arrived at a contradiction. So our assumption was wrong, and it now follows that $n + 11$ is even for every odd integer n.

18. Proof: [Here we provide a direct proof.] Since m, n are perfect squares, we may write $m = a^2$ and $n = b^2$, where a, b are (positive) integers. Then by the associative and commutative laws of multiplication for integers we find that $mn = (a^2)(b^2) = (aa)(bb) = ((aa)b)b = (a(ab))b = ((ab)a)b = (ab)(ab) = (ab)^2$, so mn is also a perfect square.

19. This result is not true, in general. For example, $m = 4 = 2^2$ and $n = 1 = 1^2$ are two positive integers that are perfect squares, but $m + n = 2^2 + 1^2 = 5$ is not a perfect square.

20. Let $m = 9 = 3^2$ and $n = 16 = 4^2$. Then $m + n = 25 = 5^2$, so the result is true.

21. Proof: We shall prove the given result by establishing the truth of its (logically equivalent) contrapositive.
Let us consider the negation of the conclusion — that is, $x < 50$ and $y < 50$. Then with $x < 50$ and $y < 50$ it follows that $x + y < 50 + 50 = 100$, and we have the negation of the hypothesis. The given result now follows by this indirect method of proof (by the contrapositive).

22. Proof: Since $4n + 7 = 4n + 6 + 1 = 2(2n + 3) + 1$, it follows from Definition 2.8 that $4n + 7$ is odd.

23. Proof: If n is odd, then $n = 2k + 1$ for some (particular) integer k. Then $7n + 8 = 7(2k + 1) + 8 = 14k + 7 + 8 = 14k + 15 = 14k + 14 + 1 = 2(7k + 7) + 1$. It then follows from Definition 2.8 that $7n + 8$ is odd.

To establish the converse, suppose that n is not odd. Then n is even, so we can write $n = 2t$, for some (particular) integer t. But then $7n + 8 = 7(2t) + 8 = 14t + 8 = 2(7t + 4)$, so it follows from Definition 2.8 that $7n+8$ is even – that is, $7n+8$ is not odd. Consequently, the converse follows by contraposition.

24. Proof: If n is even, then $n = 2k$ for some (particular) integer k. Then $31n + 12 = 31(2k) + 12 = 62k + 12 = 2(31k + 6)$, so it follows from Definition 2.8 that $31n + 12$ is even.

Conversely, suppose that n is not even. Then n is odd, so $n = 2t + 1$ for some (particular) integer t. Therefore, $31n + 12 = 31(2t + 1) + 12 = 62t + 31 + 12 = 62t + 43 = 2(31t + 21) + 1$, so from Definition 2.8 we have $31n + 12$ odd – hence, not even. Consequently, the converse follows by contraposition.

Supplementary Exercises

1.

p	q	r	s	$q \wedge r$	$\neg(s \vee r)$	$\overbrace{[(q \wedge r) \rightarrow \neg(s \vee r)]}^{t}$	$p \leftrightarrow t$
0	0	0	0	0	1	1	0
0	0	0	1	0	0	1	0
0	0	1	0	0	0	1	0
0	0	1	1	0	0	1	0
0	1	0	0	0	1	1	0
0	1	0	1	0	0	1	0
0	1	1	0	1	0	0	1
0	1	1	1	1	0	0	1
1	0	0	0	0	1	1	1
1	0	0	1	0	0	1	1
1	0	1	0	0	0	1	1
1	0	1	1	0	0	1	1
1	1	0	0	0	1	1	1
1	1	0	1	0	0	1	1
1	1	1	0	1	0	0	0
1	1	1	1	1	0	0	0

2. (a)

p	q	r	$p \rightarrow q$	$\neg p \rightarrow r$	$(p \rightarrow q) \wedge (\neg p \rightarrow r)$
0	0	0	1	0	0
0	0	1	1	1	1
0	1	0	1	0	0
0	1	1	1	1	1
1	0	0	0	1	0
1	0	1	0	1	0
1	1	0	1	1	1
1	1	1	1	1	1

(b) If p, then q, else r.

3. (a)

p	q	r	$q \leftrightarrow r$	$p \leftrightarrow (q \leftrightarrow r)$	$(p \leftrightarrow q)$	$(p \leftrightarrow q) \leftrightarrow r$
0	0	0	1	0	1	0
0	0	1	0	1	1	1
0	1	0	0	1	0	1
0	1	1	1	0	0	0
1	0	0	1	1	0	1
1	0	1	0	0	0	0
1	1	0	0	0	1	0
1	1	1	1	1	1	1

It follows from the results in columns 5 and 7 that $[p \leftrightarrow (q \leftrightarrow r)] \Leftrightarrow [(p \leftrightarrow q) \leftrightarrow r]$.

(b) The truth value assignments $p : 0;\ q : 0;\ r : 0$ result in the truth value 1 for $[p \rightarrow (q \rightarrow r)]$ and 0 for $[(p \rightarrow q) \rightarrow r]$. Consequently, these statements are not logically equivalent.

4. $p \leftrightarrow q \Longleftrightarrow (p \rightarrow q) \wedge (q \rightarrow p) \Longleftrightarrow (\neg p \vee q) \wedge (\neg q \vee p)$, so $\neg(p \leftrightarrow q) \Longleftrightarrow$
$\neg(\neg p \vee q) \vee \neg(\neg q \vee p) \Longleftrightarrow (p \wedge \neg q) \vee (q \wedge \neg p)$

5. Since $p \vee \neg q \Leftrightarrow \neg\neg p \vee \neg q \Leftrightarrow \neg p \rightarrow \neg q$, we can express the given statement as:

(1) If Kaylyn does not practice her piano lessons, then she cannot go to the movies.

But $p \vee \neg q \Leftrightarrow \neg q \vee p \Leftrightarrow q \rightarrow p$, so we can also express the given statement as:
(2) If Kaylyn is to go to the movies, then she will have to practice her piano lessons.

6. a) $p \rightarrow (q \wedge r)$
Converse: $(q \wedge r) \rightarrow p$
Inverse: $[\neg p \rightarrow \neg(q \wedge r)] \Leftrightarrow [\neg p \rightarrow (\neg q \vee \neg r)]$
Contrapositive: $[\neg(q \wedge r) \rightarrow \neg p] \Leftrightarrow [(\neg q \vee \neg r) \rightarrow \neg p]$

b) $(p \vee q) \rightarrow r$
Converse: $r \rightarrow (p \vee q)$
Inverse: $[\neg(p \vee q) \rightarrow \neg r] \Leftrightarrow [(\neg p \wedge \neg q) \rightarrow \neg r]$
Contrapositive: $[\neg r \rightarrow \neg(p \vee q)] \Leftrightarrow [\neg r \rightarrow (\neg p \wedge \neg q)]$

7.

(a) $(\neg p \vee \neg q) \wedge (F_0 \vee p) \wedge p$
(b) $(\neg p \vee \neg q) \wedge (F_0 \vee p) \wedge p$

$\Longleftrightarrow$	$(\neg p \vee \neg q) \wedge (p \wedge p)$	$F_0 \vee p \Longleftrightarrow p$
$\Longleftrightarrow$	$(\neg p \vee \neg q) \wedge p$	Idempotent Law of $\wedge$
$\Longleftrightarrow$	$p \wedge (\neg p \vee \neg q)$	Commutative Law of $\wedge$
$\Longleftrightarrow$	$(p \wedge \neg p) \vee (p \wedge \neg q)$	Distributive Law of $\wedge$ over $\vee$
$\Longleftrightarrow$	$F_0 \vee (p \wedge \neg q)$	$p \wedge \neg p \Longleftrightarrow F_0$
$\Longleftrightarrow$	$p \wedge \neg q$	F_0 is the identity for $\vee$.

8. (a) $(p \wedge \neg q) \vee (\neg r \wedge s)$
(b) Since $p \rightarrow (q \wedge \neg r \wedge s) \Leftrightarrow \neg p \vee (q \wedge \neg r \wedge s)$ it follows that $[p \rightarrow (q \wedge \neg r \wedge s)]^d \Leftrightarrow \neg p \wedge (q \vee \neg r \vee s)$.
(c) $[(p \wedge F_0) \vee (q \wedge T_0)] \wedge [r \vee s \vee F_0]$

9.

(a) contrapositive	(b) inverse	(c) contrapositive
(d) inverse	(e) converse	

10. Proof by Contradiction

(1)	$\neg(p \rightarrow s)$	Premise (Negation of Conclusion)
(2)	$p \wedge \neg s$	Step (1), $(p \rightarrow s) \iff \neg p \vee s$, DeMorgan's Laws, and the Law of Double Negation
(3)	p	Step (2) and the Rule of Conjunctive Simplification
(4)	$p \rightarrow q$	Premise
(5)	q	Steps (3), (4), and the Rule of Detachment
(6)	r	Premise
(7)	$q \wedge r$	Steps (5), (6), and the Rule of Conjunction
(8)	$(q \wedge r) \rightarrow s$	Premise
(9)	s	Steps (7), (8), and the Rule of Detachment
(10)	$\neg s$	Step (2) and the Rule of Conjunctive Simplification
(11)	$s \wedge \neg s \ (\iff F_0)$	Steps (9), (10), and the Rule of Conjunction
(12)	$\therefore p \rightarrow s$	Steps (1), (11), and the Method of Proof by Contradiction

Method 2

(1)	$(q \wedge r) \rightarrow s$	Premise
(2)	$r \rightarrow (q \rightarrow s)$	$r \rightarrow (q \rightarrow s) \iff (q \wedge r) \rightarrow s$
(3)	r	Premise
(4)	$q \rightarrow s$	Steps (2), (3), and Modus Ponens
(5)	$p \rightarrow q$	Premise
(6)	$\therefore p \rightarrow s$	Steps (4), (5), and the Law of the Syllogism

Method 3

(1)	$(q \wedge r) \to s$	Premise
(2)	$\neg s \to \neg(q \wedge r)$	Step (1) and for primitive statements u, v $u \to v \Leftrightarrow \neg v \to \neg u$ – and the 1st Substitution Rule.
(3)	$s \vee \neg(q \wedge r)$	Step (2) and for primitive statements $u, v, u \to v \Longleftrightarrow \neg u \vee v$ – and the lst Substitution Rule. Also, $\neg\neg s \Longleftrightarrow s$.
(4)	$(s \vee \neg q) \vee \neg r$	Step (3), DeMorgan's Law, and the Associative Law of $\vee$
(5)	r	Premise
(6)	$s \vee \neg q$	Steps (4), (5), and the Law of Disjunctive Syllogism
(7)	$q \to s$	Step (6) and $s \vee \neg q \Longleftrightarrow \neg q \vee s \Longleftrightarrow q \to s$
(8)	$p \to q$	Premise
(9)	$\therefore p \to s$	Steps (7), (8), and the Law of the Syllogism

Method 4 (Here we assume p as an additional premise and obtain s as our conclusion.)

(1)	p	Premise (assumed)
(2)	$p \to q$	Premise
(3)	q	Steps (1), (2), and Modus Ponens
(4)	r	Premise
(5)	$q \wedge r$	Steps (3), (4), and the Rule of Conjunction
(6)	$(q \wedge r) \to s$	Premise
(7)	$\therefore s$	Steps (5), (6), and Modus Ponens

11. (a)

p	q	r	$p \veebar q$	$(p \veebar q) \veebar r$	$q \veebar r$	$p \veebar (q \veebar r)$
0	0	0	0	0	0	0
0	0	1	0	1	1	1
0	1	0	1	1	1	1
0	1	1	1	0	0	0
1	0	0	1	1	0	1
1	0	1	1	0	1	0
1	1	0	0	0	1	0
1	1	1	0	1	0	1

It follows from the results in columns 5 and 7 that $[(p \veebar q) \veebar r] \Longleftrightarrow [p \veebar (q \veebar r)]$.
(b) The given statements are not logically equivalent. The truth value assignments $p: 1;\ q: 0;\ r: 0$ provide a counterexample.

12. p : The temperature is cool on Friday.
q : Craig wears his suede jacket.
r : The pockets (of the suede jacket) are mended.

$$\begin{array}{l} p \rightarrow (r \rightarrow q) \\ \underline{p \wedge \neg r} \\ \therefore \neg q \end{array}$$

The argument is invalid. The truth value assignments $p : 1$; $q : 1$; $r : 0$ provide a counterexample.

13.

(a)	True	(b)	False	(c)	True	(d)	True
(e)	False	(f)	False	(g)	False	(h)	True

14. a) This statement is true. Note that $1 = 7(-2) + 5(3)$, so for each integer x, $x = 7(-2x) + 5(3x)$.
b) Since 2 divides both 4 and 6, it follows that 2 divides $4y + 6z$. Consequently, the result is false for each odd integer x. [Since $2 = 4(-1) + 6(1)$, the result is true for each even integer x.]

15. Suppose that the 62 squares in this 8×8 chessboard (with two opposite missing corners) can be covered with 31 dominos. We agree to place each domino on the board so that the blue part is on top of a blue square (and the white part is then necessarily above a white square). The given chessboard contains 30 blue squares and 32 white ones. Each domino covers one blue and one white square – for a total of 31 blue squares and 31 white ones. This contradiction tells us that we cannot cover this 62 square chessboard with the 31 dominos.

16. Suppose that the 60 squares in the 8×8 chessboard (with two squares – one blue and one white – removed from each of two opposite corners) can be covered with 15 of these T-shaped figures. When covering the chessboard we agree to place each T-shaped figure on the board so that the color of each square in the T-shaped figure matches the color of the chessboard square that it covers. Let n be the number of T-shaped figures with three blue squares (and one white one) used in the covering. The chessboard contains 30 blue squares, so it follows that

$$3n + 1 \cdot (15n - n) = 30.$$

Consequently, $2n = 15$ – so 15 is both odd and even. This contradiction tells us that we cannot cover the given chessboard with these T-shaped figures.

CHAPTER 3
SET THEORY

Section 3.1

1. They are all the same set.

2. All of the statements are true except for part (f).

3. All of the statements are true except for parts (b) and (d).

4. All of the statements are true except for parts (a) and (b).

5. (a) $\{0,2\}$
(b) $\{2, 2\frac{1}{2}, 3\frac{1}{3}, 5\frac{1}{5}, 7\frac{1}{7}\}$
(c) $\{0, 2, 12, 36, 80\}$

6.

(a) True	(b) True	(c) True
(d) False	(e) True	(f) False

7. (a) $\forall x\,[x \in A \rightarrow x \in B] \wedge \exists x\,[x \in B \wedge x \notin A]$
(b) $\exists x\,[x \in A \wedge x \notin B] \vee \forall x\,[x \notin B \vee x \in A]$

8. (a) $2^7 = 128$ (b) $128 - 1 = 127$ (We substract 1 for $\emptyset$).
(c) $128 - 1 = 127$ (We subtract 1 for A) (d) 126 (e) $\binom{7}{3} = 35$
(f) For each of the other five elements of A there are two choices: Include it with 1,2 in a subset or exclude it from a subset that contains 1,2. By the rule of product there are 2^5 subsets containing 1,2.
(g) $\binom{5}{3}$ (h) $\binom{7}{0} + \binom{7}{2} + \binom{7}{4} + \binom{7}{6} = 64$ (i) $\binom{7}{1} + \binom{7}{3} + \binom{7}{6} + \binom{7}{7} = 64$

9. (a) $|A| = 6$ (b) $|B| = 7$
(c) If B has 2^n subsets of odd cardinality, then $|B| = n + 1$.

10. The only nonempty sets are in parts (d) and (f).

11. (a) There are $2^5 - 1 = 31$ nonempty subsets for the set consisting of one penny, one nickel, one dime, one quarter and one half-dollar.
(b) 30 (c) 28

12. (a) $\binom{12}{6} = 924$ (b) $\binom{6}{4}\binom{6}{2} = 225$ (c) $2^5 - 1 = 63$

13. (a) $\binom{30}{5}$
(b) Since the smallest element in A is 5 we must select the other four elements in A from $\{6, 7, 8, \ldots, 29, 30\}$. This can be done in $\binom{25}{4}$ ways.
(c) Let x denote the smallest element in A. Then there are four cases to consider.
($x = 1$) Here we can choose the other four elements in $\binom{29}{4}$ ways.
($x = 2$) Here there are $\binom{28}{4}$ selections.
($x = 3$) There are $\binom{27}{4}$ subsets possible here.
($x = 4$) In this last case we have $\binom{26}{4}$ possibilities.
In total there are $\binom{29}{4} + \binom{28}{4} + \binom{27}{4} + \binom{26}{4}$ subsets A where $|A| = 5$ and the smallest element in A is less than 5.

14. (a) There are 2^{11} subsets for $\{1, 2, 3, \ldots, 11\}$, and 2^6 subsets for $\{1, 3, 5, 7, 9, 11\}$. The 2^6 subsets of $\{1, 3, 5, 7, 9, 11\}$ contain none of the even integers 2,4,6,8,10. Hence, there are $2^{11} - 2^6 = 1984$ subsets of $\{1, 2, 3, \ldots, 11\}$ that contain at least one even integer.
(b) $2^{12} - 2^6 = 4032$
(c) For $n = 2k + 1$, where $k \geq 0$, the number of subsets of $\{1, 2, 3, \ldots, n\}$ containing at least one even integer is $2^n - 2^{k+1}$.
For $n = 2k$, with $k \geq 1$, the number of such subsets is $2^n - 2^k$.

15. Let $W = \{1\}$, $X = \{\{1\}, 2\}$, $Y = \{X, 3\}$.

16.

$(n = 6)$			1		6		15		20		15		6		1		
$(n = 7)$		1		7		21		35		35		21		7		1	
$(n = 8)$	1		8		28		56		70		56		28		8		1

17. (a) Let $x \in A$. Since $A \subseteq B$, $x \in B$. Then with $B \subseteq C$, $x \in C$. So $x \in A \Longrightarrow x \in C$ and $A \subseteq C$.
(b) Since $A \subset B \Longrightarrow A \subseteq B$, by part (a), $A \subseteq C$. With $A \subset B$, there is an element $x \in B$ such that $x \notin A$. Since $B \subseteq C$, $x \in B \Longrightarrow x \in C$, so there is an element $x \in C$ with $x \notin A$ and $A \subset C$.
(c) Since $B \subset C$ it follows that $B \subseteq C$, so by part (a) we have $A \subseteq C$. Also, $B \subset C \Longrightarrow \exists x \in \mathcal{U}\,(x \in C \wedge x \notin B)$. Since $A \subseteq B$, $x \notin B \Longrightarrow x \notin A$. So $A \subseteq C$ and $\exists x \in \mathcal{U}\,(x \in C \wedge x \notin A)$. Hence $A \subset C$.
(d) Since $A \subset B \Longrightarrow A \subseteq B$, the result follows from part (c).

18. False. Let $A = \{1\}$, $B = \{1, 2\}$, and $C = \{1, 3\}$.

19. (a) For $n, k \in \mathbf{Z}^+$ with $n \geq k+1$, consider the hexagon centered at $\binom{n}{k}$. This has the form

$$\begin{array}{ccccc} & \binom{n-1}{k-1} & & \binom{n-1}{k} & \\ \binom{n}{k-1} & & \binom{n}{k} & & \binom{n}{k+1} \\ & \binom{n+1}{k} & & \binom{n+1}{k+1} & \end{array}$$

where the two alternating triples – namely, $\binom{n-1}{k-1}, \binom{n}{k+1}, \binom{n+1}{k}$ and $\binom{n-1}{k}, \binom{n+1}{k+1}, \binom{n}{k-1}$ – satisfy $\binom{n-1}{k-1}\binom{n}{k+1}\binom{n+1}{k} = \binom{n-1}{k}\binom{n+1}{k+1}\binom{n}{k-1}$.

(b) For $n, k \in \mathbf{Z}^+$ with $n \geq k+1$,

$$\binom{n-1}{k-1}\binom{n}{k+1}\binom{n+1}{k} = \left[\frac{(n-1)!}{(k-1)!(n-k)!}\right]\left[\frac{n!}{(k+1)!(n-k-1)!}\right]\left[\frac{(n+1)!}{k!(n+1-k)!}\right]$$

$$= \left[\frac{(n-1)!}{k!(n-1-k)!}\right]\left[\frac{(n+1)!}{(k+1)!(n-k)!}\right]\left[\frac{n!}{(k-1)!(n-k+1)!}\right] =$$

$$\binom{n-1}{k}\binom{n+1}{k+1}\binom{n}{k-1}.$$

20. (a) Each of these strictly increasing sequences of integers corresponds with a subset of $\{2,3,4,5,6\}$. Therefore there are 2^5 such strictly increasing sequences.

(b) 2^5

(c) 2^{35} and 2^{35}

(d) Let m, n be positive integers with $m < n$. The number of strictly increasing sequences of integers that start with m and end with n is $2^{[(n-m)+1]-2} = 2^{n-m-1}$.

21. $(1/4)\binom{n}{5} = \binom{n-1}{4} \Longrightarrow (1/4)[(n!)/(5!(n-5)!)] = (n-1)!/(4!(n-5)!) \Longrightarrow n! = 20(n-1)! \Longrightarrow n = 20$.

22. a) $2n$ b) $4n = 2^2 n$ c) $2^k n$

23. For a given $n \in \mathbf{N}$, we need to find $k \in \mathbf{N}$ so that the three consecutive entries $\binom{n}{k}$, $\binom{n}{k+1}$, $\binom{n}{k+2}$ are in the ratio $1:2:3$. [Consequently, $n \geq 2$ (and $k \geq 0$).] In order to obtain the given ratio we must have

$$\binom{n}{k+1} = 2\binom{n}{k} \quad \text{and} \quad \binom{n}{k+2} = 3\binom{n}{k}.$$

From $\binom{n}{k+1} = 2\binom{n}{k}$, it follows that $2\frac{n!}{(k!)(n-k)!} = \frac{n!}{(k+1)!(n-k-1)!}$ so $2k+2 = n-k$, or $n = 2+3k$. Likewise, $\binom{n}{k+2} = 3\binom{n}{k}$ implies that $3\frac{n!}{(k!)(n-k)!} = \frac{n!}{(k+2)!(n-k-2)!}$, and we find that $3(k+2)(k+1) = (n-k)(n-k-1)$. Consequently, with $n = 2+3k$, we have $3(k+2)(k+1) =$

$(2+2k)(1+2k)$, or $0 = k^2 - 3k - 4 = (k-4)(k+1)$. Since $k \geq 0$, it follows that $k = 4$ and $n = 14$. So the 5th, 6th, and 7th entries in the row for $n = 14$ provide the unique solution.

24.

0000	$\emptyset$	0011	$\{y, z\}$
1000	$\{w\}$	1011	$\{w, y, z\}$
1100	$\{w, x\}$	1111	$\{w, x, y, z\}$
0100	$\{x\}$	0111	$\{x, y, z\}$
0110	$\{x, y\}$	0101	$\{x, z\}$
1110	$\{w, x, y\}$	1101	$\{w, x, z\}$
1010	$\{w, y\}$	1001	$\{w, z\}$
0010	$\{y\}$	0001	$\{z\}$

25. As an ordered set, $A = \{x, v, w, z, y\}$.

26. $\binom{n}{0} + \binom{n+1}{1} + \binom{n+2}{2} + \binom{n+3}{3} + \ldots + \binom{n+r-1}{r-1} + \binom{n+r}{r} = \binom{n+1}{0} + \binom{n+1}{1} + \binom{n+2}{2} + \binom{n+3}{3} + \ldots + \binom{n+r-1}{r-1} + \binom{n+r}{r} = \binom{n+2}{1} + \binom{n+2}{2} + \binom{n+3}{3} + \ldots + \binom{n+r-1}{r-1} + \binom{n+r}{r} = \binom{n+3}{2} + \binom{n+3}{3} + \ldots + \binom{n+r-1}{r-1} + \binom{n+r}{r} = \binom{n+4}{3} + \ldots + \binom{n+r-1}{r-1} + \binom{n+r}{r} = \ldots = \binom{n+r}{r-1} + \binom{n+r}{r} = \binom{n+r-1}{r}$

27. (a) If $S \in S$, then since $S = \{A | A \notin A\}$ we have $S \notin S$.
(b) If $S \notin S$, then by the definition of S it follows that $S \in S$.

28. (b)

```
10  Random
20  Dim B(12), S(6)
30  B(1) = 2: B(2) = 3: B(3) = 5: B(4) = 7
40  B(5) = 11: B(6) = 13: B(7) = 17: B(8) = 19
50  B(9) = 23: B(10) = 29: B(11) = 31: B(12) = 37
60  For I = 1 To 6
70        S(I) = Int(Rnd*40) + 1
80        For J = 1 To I - 1
90              If S(J) = S(I) Then GOTO 70
100       Next J
110 Next I
120 For I = 1 To 6
130       For J = 1 To 12
140             If S(I) = B(J) Then GOTO 170
150       Next J
160       GOTO 240
170 Next I
180 Print "The subset S contains the elements";
```

```
190 For I = 1 To 5
200       Print S(I); ", ";
210 Next I
220 Print S(6); " and is a subset of B"
230 GOTO 290
240 Print "The subset S contains the elements";
250 For I = 1 To 5
260       Print S(I); ", ";
270 Next I
280 Print S(6); " but it is not a subset of B"
290 End
```

29.

```
procedure Subsets4(i,j,k,l: positive integers)
begin
   for i := 1 to 4 do
      for j := i+1 to 5 do
         for k := j+1 to 6 do
            for l := k+1 to 7 do
               print ({i,j,k,l})
end
```

30.

```
Program List_subsets4 (Input, Output);
Const
   Size = 10;
Type
   Member_type = 1..Size;
   Set_type = set of Member_type;
Var
   n: 1..Size;
   S: Set_type;

Procedure Write_set (S: Set_type);
Var
   i: 1..Size;
Begin
   Write ('{');
   For i := 1 to Size do
```

```
        If i in S then
            Begin
                S := S - [i];
                If S <> [ ], then
                    Write (i:3, ',')
                Else Write (i: 3);
            End;
    Writeln ('}');
End;

Procedure Subsets (L,R : Set_type; i: Member_type);
Begin
    If i <= n then
        Begin
            Subsets (L + [i], R, i+1);
            Subsets (L, R + [i], i+1);
        End
    Else
        Begin
            Write_set (L);
            Write_set (R);
        End;
End;

Begin
    Write ('What is the value of n?');
    Readln (n);
    Subsets ([1],[ ],2);
End.
```

Section 3.2

1.

(a) {1,2,3,5}	(b) A	(c) $\mathcal{U}-\{2\}$
(d) $\mathcal{U}-\{2\}$	(e) {4,8}	(f) {1,2,3,4,5,8}
(g) $\emptyset$	(h) {2,4,8}	(i) {1,3,4,5,8}

2.

(a) [2,3]	(b) [0,7)	(c) $(-\infty, 0) \cup (3, +\infty)$
(d) $[0,2) \cup (3,7)$	(e) [0,2)	(f) (3,7)

3. (a) Since $A = (A - B) \cup (A \cap B)$ we have $A = \{1, 3, 4, 7, 9, 11\}$. Similarly we find $B = \{2, 4, 6, 8, 9\}$.

(b) $C = \{1,2,4,5,9\}$, $D = \{5,7,8,9\}$.

4.

(a) (i) True (ii) False (iii) False
(iv) True (v) True (vi) False

(b) (i) E (ii) B (iii) D
(iv) D (v) $\mathbf{Z} - A = \{2n+1 | n \in \mathbf{Z}\}$ = The set of all (positive and negative) odd integers (vi) E

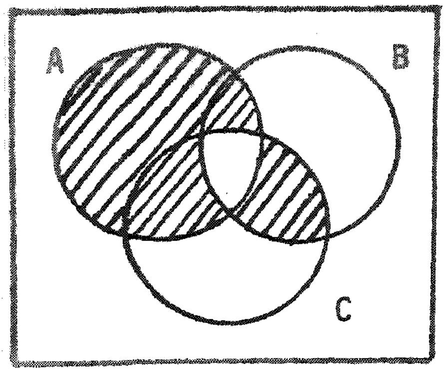
5.

(a)	True	(b)	True	(c)	True	(d)	False	(e)	True
(f)	True	(g)	True	(h)	False	(i)	False		

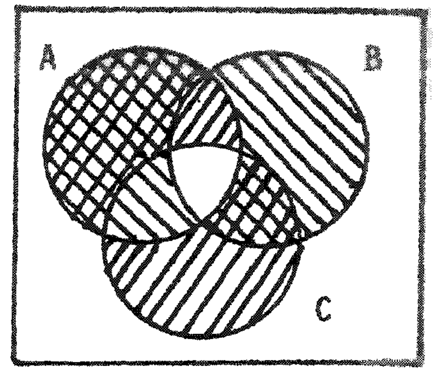
6. (a) $x \in A \cap C \Longrightarrow (x \in A \text{ and } x \in C) \Longrightarrow (x \in B \text{ and } x \in D)$, since $A \subseteq B$ and $C \subseteq D \Longrightarrow x \in B \cap D$, so $A \cap C \subseteq B \cap D$.
$x \in A \cup C \Longrightarrow x \in A$ or $x \in C$. If $x \in A$, then $x \in B$, since $A \subseteq B$. Likewise, $x \in C \Longrightarrow x \in D$. In either case, $x \in A \cup C \Longrightarrow x \in B \cup D$, so $A \cup C \subseteq B \cup D$.
(b) Let $A \subseteq B$. We always have $\emptyset \subseteq A \cap \overline{B}$, so let $x \in A \cap \overline{B}$. Then $x \in A$ and $x \in \overline{B}$. $x \in A \Longrightarrow x \in B$, since $A \subseteq B$. $x \in B$, $x \in \overline{B} \Longrightarrow x \in B \cap \overline{B} = \emptyset$, so $A \cap \overline{B} = \emptyset$. Conversely, for $A \cap \overline{B} = \emptyset$, let $x \in A$. If $x \notin B$, then $x \in \overline{B}$, so $x \in A \cap \overline{B} = \emptyset$. Hence $x \in B$ and $A \subseteq B$.
(c) Follows from part (b) by the principle of duality.

7. (a) False. Let $\mathcal{U} = \{1,2,3\}$, $A = \{1\}$, $B = \{2\}$, $C = \{3\}$. Then $A \cap C = B \cap C$ but $A \neq B$.
(b) False. Let $\mathcal{U} = \{1,2\}$, $A = \{1\}$, $B = \{2\}$, $C = \{1,2\}$. Then $A \cup B = A \cup C$ but $A \neq B$.
(c) $x \in A \Longrightarrow x \in A \cup C \Longrightarrow x \in B \cup C$. So $x \in B$ or $x \in C$. If $x \in B$, then we are finished. If $x \in C$, then $x \in A \cap C = B \cap C$ and $x \in B$. In either case, $x \in B$ so $A \subseteq B$. Likewise, $y \in B \Longrightarrow y \in B \cup C = A \cup C$, so $y \in A$ or $y \in C$. If $y \in C$, then $y \in B \cap C = A \cap C$. In either case, $y \in A$ and $B \subseteq A$. Hence $A = B$.
(d) Let $x \in A$. Consider two cases: (i) $x \in C \Longrightarrow x \notin A\Delta C \Longrightarrow x \notin B\Delta C \Longrightarrow x \in B$. (ii) $x \notin C \Longrightarrow x \in A\Delta C \Longrightarrow x \notin B\Delta C \Longrightarrow x \in B$. In either case $A \subseteq B$. In a similar way we find $B \subseteq A$, so $A = B$.

8. (a)



$A\Delta(B \cap C)$



$(A\Delta B) \cap (A\Delta C)$

From the Venn diagrams it follows that $A\Delta(B \cap C) \neq (A\Delta B) \cap (A\Delta C)$, so the result is false.

(b) True (c) True

9. $(A \cap B) \cup C = \{d, x, z\}$ which has $2^3 - 1 = 7$ proper subsets; $A \cap (B \cup C) = \{d\}$ which has 1 proper subset.

10. (a) 0 (b) 0 and 1

11. (a) $\emptyset = (A \cup B) \cap (A \cup \overline{B}) \cap (\overline{A} \cup B) \cap (\overline{A} \cup \overline{B})$
(b) $A = A \cup (A \cap B)$
(c) $A \cap B = (A \cup B) \cap (A \cup \overline{B}) \cap (\overline{A} \cup B)$
(d) $A = (A \cap B) \cup (A \cap \mathcal{U})$

12. The dual of the statement $A \cap B = A$ is the statement $A \cup B = A$. But $A \cup B = A \iff B \subseteq A$, so the dual of the statement $A \subseteq B$ is the statement $B \subseteq A$.

13. (a) False. Let $\mathcal{U} = \{1,2,3\}$, $A = \{1\}$, $B = \{2\}$. $P(A) = \{\emptyset, A\}, P(B) = \{\emptyset, B\}$, $P(A \cup B) = \{\emptyset, \{1\}, \{2\}, \{1,2\}\}$, and $\{1,2\} \notin P(A) \cup P(B)$.
(b) $X \in P(A) \cap P(B) \iff X \in P(A)$ and $X \in P(B) \iff X \subseteq A$ and $X \subseteq B \iff X \subseteq A \cap B \iff X \in P(A \cap B)$, so $P(A) \cap P(B) = P(A \cap B)$.

14. (a) & (c)

A	B	$A \cap B$	$\overline{A \cap B}$	$\overline{A} \cup \overline{B}$	$A \cup (A \cap B)$
0	0	0	1	1	0
0	1	0	1	1	0
1	0	0	1	1	1
1	1	1	0	0	1

(b)

A	$A \cup A$
0	0
1	1

(d)

A	B	C	$A \cap B$	$\overline{A} \cap C$	$\overline{(A \cap B) \cup (\overline{A} \cap C)}$
0	0	0	0	0	1
0	0	1	0	1	0
0	1	0	0	0	1
0	1	1	0	1	0
1	0	0	0	0	1
1	0	1	0	0	1
1	1	0	1	0	0
1	1	1	1	0	0

$A \cap \overline{B}$	$\overline{A} \cap \overline{C}$	$(A \cap \overline{B}) \cup (\overline{A} \cap \overline{C})$
0	1	1
0	0	0
0	1	1
0	0	0
1	0	1
1	0	1
0	0	0
0	0	0

15. (a) $2^6 = 64$ (b) 2^n

(c) In the columns for A, B, whenever a 1 occurs in the column for A, a 1 likewise occurs in the same position in the column for B.

(d)

A	B	C	$A\cup\overline{B}$	$A\cap B$	$\overline{B\cap C}$	$(A\cap B)\cup(\overline{B\cap C})$
0	0	0	1	0	1	1
0	0	1	1	0	1	1
0	1	0	0	0	1	1
0	1	1	0	0	0	0
1	0	0	1	0	1	1
1	0	1	1	0	1	1
1	1	0	1	1	1	1
1	1	1	1	1	0	1

16.

	Steps	**Reasons**
	$(A\cap B)\cup[B\cap((C\cap D)\cup(C\cap\overline{D}))]$	
$=$	$(A\cap B)\cup[B\cap(C\cap(D\cup\overline{D}))]$	Distributive Law of $\cap$ over $\cup$
$=$	$(A\cap B)\cup[B\cap(C\cap\mathcal{U})]$	$D\cup\overline{D}=\mathcal{U}$
$=$	$(A\cap B)\cup(B\cap C)$	Identity Law $[C\cap\mathcal{U}=C]$
$=$	$(B\cap A)\cup(B\cap C)$	Commutative Law of $\cap$
$=$	$B\cap(A\cup C)$	Distributive Law of $\cap$ over $\cup$

17. (a) $A\cap(B-A)=A\cap(B\cap\overline{A})=B\cap(A\cap\overline{A})=B\cap\emptyset=\emptyset$

(b) $[(A\cap B)\cup(A\cap B\cap\overline{C}\cap D)]\cup(\overline{A}\cap B)=(A\cap B)\cup(\overline{A}\cap B)$ by the Absorption Law $=(A\cup\overline{A})\cap B=\mathcal{U}\cap B=B$

(c) $(A-B)\cup(A\cap B)=(A\cap\overline{B})\cup(A\cap B)=A\cap(\overline{B}\cup B)=A\cap\mathcal{U}=A$

(d) $\overline{A}\cup\overline{B}\cup(A\cap B\cap\overline{C})=\overline{(A\cap B)}\cup[(A\cap B)\cap\overline{C}]=[\overline{(A\cap B)}\cup(A\cap B)]\cap[\overline{(A\cap B)}\cup\overline{C}]=[\overline{(A\cap B)}\cup\overline{C}]=\overline{A}\cup\overline{B}\cup\overline{C}$

18. $\bigcup_{n=1}^{7}A_n=A_7=\{1,2,3,4,5,6,7\}$, $\bigcap_{n=1}^{7}A_n=A_1=\{1\}$.

$\bigcup_{n=1}^{m}A_n=A_m=\{1,2,3,\ldots,m-1,m\}$, $\bigcap_{n=1}^{m}A_n=A_1=\{1\}$.

19.

(a) $[-6,9]$ (b) $[-8,12]$ (c) $\emptyset$ (d) $[-8,-6)\cup(9,12]$
(e) $[-14,21]$ (f) $[-2,3]$ (g) $\mathbf{R}$ (h) $[-2,3]$

20. $x\in\overline{\bigcap_{i\in I}A_i}\iff x\notin\bigcap_{i\in I}A_i\iff x\notin A_i$ for at least one $i\in I\iff x\in\overline{A_i}$ for at least one $i\in I\iff x\in\bigcup_{i\in I}\overline{A_i}$.

Section 3.3

1. Here the universe $\mathcal{U}$ comprises the 600 freshmen. If we let $A, B \subseteq \mathcal{U}$ be the subsets
A: the freshmen who attended the first showing
B: the freshmen who attended the second showing,
then $|\mathcal{U}| = 600$, $|A| = 80$, $|B| = 125$, and $|\overline{A} \cap \overline{B}| = 450$.
Since $|\overline{A} \cap \overline{B}| = |\overline{A \cup B}| = 450$, it follows that $|A \cup B| = 600 - 450 = 150$. Consequently, $|A \cap B| = |A| + |B| - |A \cup B| = 80 + 125 - 150 = 55$ – that is, 55 of the 600 freshmen attended the movie twice.

2. Here the universe $\mathcal{U}$ comprises the 2000 automobile batteries. If we let $A, B \subseteq \mathcal{U}$ be the subsets
A: the batteries with defective terminals
B: the batteries with defective plates,
then $|\mathcal{U}| = 200$, $|\overline{A} \cap \overline{B}| = 1920$, $|B| = 60$, and $|A \cap B| = 20$.

Since $\overline{A} \cap \overline{B} = \overline{A \cup B}$, it follows that $|A \cup B| = 2000 - 1920 = 80$. From $|A \cup B| = |A| + |B| - |A \cap B|$ we learn that $|A| = |A \cup B| - |B| + |A \cap B| = 80 - 60 + 20 = 40$ – that is, 40 of her 2000 batteries have defective terminals.

3. There are 2^9 such strings that start with three 1's and 2^8 that end in four 0's. In addition, 2^5 of these strings start with three 1's and end in four 0's. Consequently, the number that start with three 1's or end in four 0's is

$$2^9 + 2^8 - 2^5 = 512 + 256 - 32 = 736.$$

4. (a) Here $A \cup B \cup C = C$, so $|A \cup B \cup C| = |C| = 5000$.
(b) Here $A \cap B \cap C = \emptyset$ as well, so it follows from the formula for $|A \cup B \cup C| = |A| + |B| + |C| = 50 + 500 + 5000 = 5550$.
(c) $|A \cup B \cup C| = |A| + |B| + |C| - |A \cap B| - |A \cup C| - |B \cap C| + |A \cap B \cap C| = 50 + 500 + 5000 - 3 - 3 - 3 + 1 = 5542$.

5. $9! + 9! - 8!$

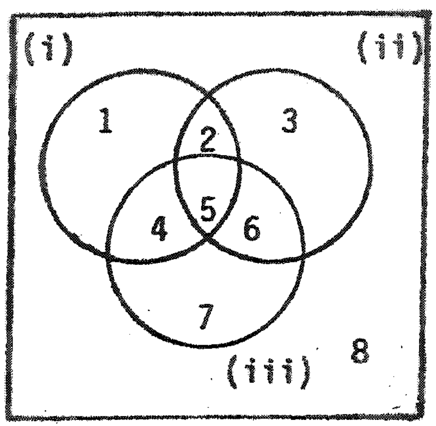
6. (a) 12 (b) 2 (c) 16

7. (a) There are 24! permutations containing each of the patterns O U T and D I G. There are 22! permutations containing both patterns. Consequently there are $2(24!) - 22!$ permutations containing either O U T or D I G.
(b) There are 26! permutations in total. Of these there are 24! that contain each of the patterns M A N and A N T and 23! that contain both patterns (i.e., contain M A N T). Hence there are $2(24!) - 23!$ permutations that contain either M A N or A N T and $26! - [2(24!) - 23!]$ permutations that contain neither pattern.

8. For the pattern F U N we consider four cases.
(a) F U N _ _ _ _. Here the blanks can be filled in $(36)^3$ ways.

(b) _ F U N _ _ _. Here the blanks can be filled in $(26)(36)^2$ ways.
(c) _ _ F U N _ _. Again there are $26(36)^2$ ways to fill in the blanks.
(d) _ _ _ F U N. There are also $26(36)^2$ ways to fill in the blanks here.
Consequently the number of six character variable names containing F U N is $(36)^3 + 3(26)(36)^2 - 1$, because the variable F U N F U N is counted in both case (a) and case (d). There are also $(36)^3 + 3(26)(36)^2 - 1$ of these variables that contain T I P and two that contain both F U N and T I P. Consequently, the number of these six character variables that contain either F U N or T I P is $2[(36)^3 + 3(26)(36)^2 - 1] - 2$.

9.



The circle labeled (i) is for the arrangements with consecutive S's; circle (ii) is for consecutive E's; and circle (iii) for consecutive L's. The answer to the problem is the number of arrangements in region 8 which we obtain as follows. For region 5 there are $10!$ ways to arrange the 10 symbols M,I,C,A,N,O,U,SS,EE,LL. For regions 2,4,6 there are $(11!/2!) - 10!$ arrangements containing exactly two pairs of consecutive letters. Finally each of regions 1,3,7 contains $(12!/(2!2!)) - 2[(11!/2!) - 10!] - 10!$ arrangements, so region 8 contains $[13!/(2!)^3] - 3[12!/(2!2!)] + 3(11!/2!) - 10!$ arrangements.

10. The number of arrangements with either H before E, or E before T, or T before M equals the total number of arrangements (i.e., $7!$) minus the number of arrangements where E is before H, and T is before E, and M is before T. There are $3!$ ways to arrange C, I, S. For each arrangement there are four locations (one at the start, two between pairs of letters, and one at the end) to select from, with repetition, to place M, T, E, H in this prescribed order. Hence there are $(3!)\binom{4+4-1}{4} = (3!)\binom{7}{4}$ arrangements where M is before T, T before E, and E before H. Consequently, there are $7! - (3!)\binom{7}{4}$ arrangements with either H before E, or E before T, or T before M.

Section 3.4

1. (a) $Pr(A) = |A|/|S| = 3/8$
(b) $Pr(B) = |B|/|S| = 4/8 = 1/2$
(c) $A \cap B = \{a, c\}$ so $Pr(A \cap B) = 2/8 = 1/4$
(d) $A \cup B = \{a, b, c, e, g\}$ so $Pr(A \cup B) = 5/8$
(e) $\overline{A} = \{d, e, f, g, h\}$ and $Pr(\overline{A}) = 5/8 = 1 - 3/8 = 1 - Pr(A)$
(f) $\overline{A} \cup B = \{a, c, d, e, f, g, h\}$ with $Pr(\overline{A} \cup B) = 7/8$
(g) $A \cap \overline{B} = \{b\}$ so $Pr(A \cap \overline{B}) = 1/8$.

2. (a) $S = \{(x, y) | x, y \in \{1, 2, 3, \ldots, 20\}\}$
(b) $S = \{(x, y) | x, y \in \{1, 2, 3, \ldots, 20\}, x \neq y\}$

3. Here each equally likely outcome has probability $\frac{1}{25} = 0.04$. Consequently, there are $\frac{0.24}{0.04} = 6$ outcomes in A.

4. The probability of each equally likely outcome is $\frac{0.14}{7} = 0.02 = \frac{1}{n}$. Therefore, $n = \frac{1}{0.02} = 50$.

5. (a) $\binom{6}{2}/\binom{12}{2} = 15/66 = 5/22 = 0.2272727\ldots$
(b) $[\binom{1}{1}\binom{10}{1} + \binom{10}{1}\binom{1}{1} + \binom{1}{1}\binom{1}{1}]/\binom{12}{2} = 21/66 = 7/22 = 1 - [\binom{10}{2}/\binom{12}{2}]$

6. $\mathcal{S} = \{\{x, y\} | x, y \in \{1, 2, 3, \ldots, 99, 100\}, x \neq y\}$
$A = \{\{x, x+1\} | x \in \{1, 2, 3, \ldots, 99\}\}$
$|\mathcal{S}| = \binom{100}{2} = 4950$; $|A| = 99$
$Pr(A) = 99/4950 = 1/50$

7. $\mathcal{S} = \{\{x, y\} | x, y \in \{1, 2, 3, \ldots, 99, 100\}, x \neq y\}$
$A = \{\{x, y\} | \{x, y\} \in \mathcal{S}, x + y \text{ is even}\}$
$= \{\{x, y\} | \{x, y\} \in \mathcal{S}, x, y \text{ even}\} \cup \{\{x, y\} | \{x, y\} \in \mathcal{S}, x, y \text{ odd}\}$
$|\mathcal{S}| = \binom{100}{2} = 4950$; $|A| = \binom{50}{2} + \binom{50}{2} = 2450$
$Pr(A) = 2450/4950 = 49/99$

8. $\mathcal{S} = \{\{a, b, c\} | a, b, c \in \{1, 2, 3, \ldots, 99, 100\},\ a \neq b,\ a \neq c,\ b \neq c\}$
$A = \{\{a, b, c\} | \{a, b, c\} \in \mathcal{S}, a + b + c \text{ is even}\} = \{\{a, b, c\} | \{a, b, c\} \in \mathcal{S},\ a, b, c$ are even, or one of a, b, c is even and the other two integers are odd$\}$
$|\mathcal{S}| = \binom{100}{3} = 161,700$; $|A| = \binom{50}{3} + \binom{50}{1}\binom{50}{2} = 19,600 + 61,250 = 80,850$
$Pr(A) = 80,850/161,700 = 1/2$.

9. The sample space $\mathcal{S} = \{(x_1, x_2, x_3, x_4, x_5, x_6) | x_i = H \text{ or } T, 1 \leq i \leq 6\}$. Hence $|\mathcal{S}| = 2^6 = 64$.
(a) Here the event $A = \{HHHHHH\}$ and $Pr(A) = 1/64$.
(b) The event $B = \{HHHHHT, HHHHTH, HHHTHH, HHTHHH, HTHHHH, THHHHH\}$ and $Pr(B) = 6/64 = 3/32$.
(c) There are $6!/(4!2!) = 15$ ways to arrange two heads and four tails, so the probability for this event is $15/64$.
(d) 0 heads: 1 arrangement
2 heads: $[6!/(2!4!)] = 15$ arrangements
4 heads: $[6!/(4!2!)] = 15$ arrangements
6 heads: 1 arrangement
The event here includes exactly 32 of the 64 arrangements in $\mathcal{S}$, so the probability for an even number of heads is $32/64 = 1/2$.
(e) 4 heads: $[6!/(4!2!)] = 15$ arrangements
5 heads: $[6!/(5!1!)] = 6$ arrangements
6 heads: 1 arrangement
Here the probability is $22/64 = 11/32$.

10. (a) $\binom{4}{1}\binom{20}{4}/\binom{25}{6} = (4 \cdot 4845)/177100 \doteq 0.10943$

(b) $\binom{14}{3}\binom{10}{2}/\binom{25}{6} = (364 \cdot 45)/177100 \doteq 0.09249$
(c) $\binom{4}{1}\binom{9}{1}\binom{10}{2}/\binom{25}{6} = (4 \cdot 9 \cdot 45)/177100 \doteq 0.00915$

11. (a) Let $S =$ the sample space $= \{(x_1, x_2, x_3) | 1 \le x_i \le 6,\ i = 1, 2, 3\}$; $|S| = 6^3 = 216$.
Let $A = \{(x_1, x_2, x_3) | x_1 < x_2 \text{ and } x_1 < x_3\} = \bigcup_{n=1}^{5} \{(n, x_2, x_3) | n < x_2 \text{ and } n < x_3\}$.
For $1 \le n \le 5$, $|\{(n, x_2, x_3) | n < x_2 \text{ and } n < x_3\}| = (6 - n)^2$.
Consequently, $|A| = 5^2 + 4^2 + 3^2 + 2^2 + 1^2 = 55$.
Therefore, $Pr(A) = 55/216$.
(b) With S as in part (a), let $B = \{(x_1, x_2, x_3) | x_1 < x_2 < x_3\}$.
Then $|\{(1, x_2, x_3) | 1 < x_2 < x_3\}| = 10$,
$|\{(2, x_2, x_3) | 2 < x_2 < x_3\}| = 6$,
$|\{(3, x_2, x_3) | 3 < x_2 < x_3\}| = 3$, and
$|\{(4, x_2, x_3) | 4 < x_2 < x_3\}| = 1$,
so $|B| = 20$ and $Pr(B) = 20/216 = 5/54$.

12. (a) 10 (b) 1 (c) 4/15

13. (a) $\frac{14!}{15!} = \frac{1}{15}$ (b) $[(14!) + (14!)]/(15!) = 2(14!)/(15!) = 2/15$
(c) $(2)(9)(13!)/(15!) = 3/35$

14. (a) $24/300 = 0.08$
(b) (i) There are 180 students who can program in Java. Two can be selected in $\binom{180}{2}$ ways. The sample space consists of the $\binom{300}{2}$ pairs of students. So the probability that two students selected at random can both program in Java is $\binom{180}{2}/\binom{300}{2} = (180)(179)/(300)(299) \doteq 0.36$. (ii) $\binom{162}{2}/\binom{300}{2} \doteq 0.29$.

15. $Pr(A) = 1/3$; $Pr(B) = 7/15$, $Pr(A \cap B) = 2/15$; $Pr(A \cup B) = 2/3$. $Pr(A \cup B) = 2/3 = (1/3) + (7/15) - (2/15) = Pr(A) + Pr(B) - Pr(A \cap B)$.

16. (a) $2[(5!/2!)]/[(7!/(2!2!))] = 120/1260 \doteq 0.0952$
(b) $[(7!/(2!2!)) - 2((6!/2!) - 5!) - 5!]/[(7!/(2!2!))] =$
$[(7!/(2!2!)) - 2(6!/2!) + 5!]/[(7!/(2!2!))] =$
$[1260 - 720 + 120]/(1260) = 660/1260 \doteq 0.5238$

Section 3.5

1. $Pr(\overline{A}) = 1 - Pr(A) = 1 - 0.4 = 0.6$
$Pr(\overline{B}) = 1 - Pr(B) = 1 - 0.3 = 0.7$
$Pr(A \cup B) = Pr(A) + Pr(B) - Pr(A \cap B) = 0.4 + 0.3 - 0.2 = 0.5$
$Pr(\overline{A \cup B}) = 1 - Pr(A \cup B) = 1 - 0.5 = 0.5$
$Pr(A \cap \overline{B}) = Pr(A) - Pr(A \cap B)$ because $A = (A \cap \overline{B}) \cup (A \cap B)$ with $(A \cap \overline{B}) \cap (A \cap B) = \emptyset$.

So $Pr(A \cap \overline{B}) = 0.4 - 0.2 = 0.2$
$Pr(\overline{A} \cap B) = Pr(B) - Pr(A \cap B) = 0.3 - 0.2 = 0.1$
$Pr(A \cup \overline{B}) = Pr(\overline{\overline{A} \cap B}) = 1 - Pr(\overline{A} \cap B) = 1 - 0.1 = 0.9$
$Pr(\overline{A} \cup B) = Pr(\overline{A \cap \overline{B}}) = 1 - Pr(A \cap \overline{B}) = 1 - 0.2 = 0.8$

2. (a) $\binom{8}{6}(\frac{1}{2})^6(\frac{1}{2})^2 = \binom{8}{6}(\frac{1}{2})^8 \doteq 0.109375$
(b) $\binom{8}{6}(\frac{1}{2})^6(\frac{1}{2})^2 + \binom{8}{7}(\frac{1}{2})^7(\frac{1}{2}) + \binom{8}{8}(\frac{1}{2})^8 = (\frac{1}{2})^8[\binom{8}{6} + \binom{8}{7} + \binom{8}{8}] \doteq 0.144531$
(c) $\binom{8}{2}(\frac{1}{2})^2(\frac{1}{2})^6 \doteq 0.109375$
(d) $\binom{8}{0}(\frac{1}{2})^8 + \binom{8}{1}(\frac{1}{2})(\frac{1}{2})^7 + \binom{8}{2}(\frac{1}{2})^2(\frac{1}{2})^6 = (\frac{1}{2})^8[\binom{8}{0} + \binom{8}{1} + \binom{8}{2}] \doteq 0.144531$

3. (a) $S = \{(x,y)|x,y \in \{1,2,3,\ldots,10\}, x \neq y\}$.
(b) For $1 \leq y \leq 9$, if y is the label on the second ball drawn, then there are $10 - y$ possible values for x so that $(x,y) \in S$ and $x > y$. Consequently, if A denotes the event described here, then $|A| = 9 + 8 + 7 + \cdots + 1 = 45$ and $Pr(A) = |A|/|S| = 45/90 = 1/2$.
(c) Let $B = \{(v,w)|v \text{ even}, w \text{ odd}\}$. Then we want $Pr(B \cup C)$ where $B \cap C = \emptyset$. So $Pr(B \cup C) = Pr(B) + Pr(C) = \frac{25}{90} + \frac{25}{90} = \frac{50}{90} = \frac{5}{9}$.

4. Here $Pr(A) = \frac{13}{52}$, $Pr(B) = \frac{26}{52}$, $Pr(C) = \frac{12}{52}$, $Pr(A \cap B) = \frac{0}{52}$, $Pr(A \cap C) = \frac{3}{52}$, $Pr(B \cap C) = \frac{16}{52}$, and $Pr(A \cap B \cap C) = \frac{0}{52}$. So

$$Pr(A \cup B \cup C) = \frac{13}{52} + \frac{26}{52} + \frac{12}{52} - \frac{0}{52} - \frac{3}{52} - \frac{6}{52} + \frac{0}{52} = \frac{42}{52} = \frac{21}{26}.$$

5. Since A, B are disjoint we know that $Pr(A \cup B) = Pr(A) + Pr(B)$, so $Pr(B) = 0.7 - 0.3 = 0.4$.

6. $Pr(A \Delta B) = Pr(A) + Pr(B) - 2Pr(A \cap B)$

7. (a) Let p be the probability for the outcome 1. Then for $1 \leq n \leq 6$, the probability for the outcome n is np and $p + 2p + 3p + 4p + 5p + 6p = 1$. Consequently $p = 1/21$.

So the probability for a 5 or 6 is $\frac{5}{21} + \frac{6}{21} = \frac{11}{21}$.
(b) The probability the outcome is even is $\frac{2}{21} + \frac{4}{21} + \frac{6}{21} = \frac{12}{21}$.
(c) $1 - \frac{12}{21} = \frac{9}{21} = \frac{1}{21} + \frac{3}{21} + \frac{5}{21}$.

8. (a) Let x be the outcome on the first die and y the outcome on the second die. Here we want $Pr(x = 6, y = 4) + Pr(x = 5, y = 5) + Pr(x = 4, y = 6))$. So the probability a 10 is rolled is $(\frac{6}{21})(\frac{4}{21}) + (\frac{5}{21})(\frac{5}{21}) + (\frac{4}{21})(\frac{6}{21}) = \frac{24+25+24}{441} = \frac{73}{441} \doteq 0.165533$.
(b) The probability of rolling an 11 is $(\frac{6}{21})(\frac{5}{21}) + (\frac{5}{21})(\frac{6}{21}) = \frac{60}{441}$. For 12 the probability is $(\frac{6}{21})(\frac{6}{21}) = \frac{36}{441}$. So the probability of rolling at least 10 is $\frac{73+60+36}{441} = \frac{169}{441} \doteq 0.383220$.
(c) $(\frac{1}{21})^2 + (\frac{2}{21})^2 + \cdots + (\frac{6}{21})^2 = \frac{1+4+9+16+25+36}{441} = \frac{91}{441} \doteq 0.206349$.

9. Here the sample space $S = \{x_1x_2x_3x_4x_5|x_i \in \{H,T\}, 1 \leq i \leq 5\}$. So $|S| = 2^5 = 32$. The event A of interest here is $A = \{HHHHH, HHHHT, HHHTH, HHHTT, HHTHT, HHTHH\}$ and $Pr(A) = 6/32 = 3/16$.

10. $Pr(A) = \frac{5+60}{125} = \frac{65}{125} = \frac{13}{25} = 0.52$ $\qquad$ $Pr(B) = \frac{60+30+15}{125} = \frac{105}{125} = \frac{21}{25} = 0.84$

$Pr(A \cap B) = \frac{60}{125} = \frac{12}{25} = 0.48$
$Pr(A \cup B) = Pr(A) + Pr(B) - Pr(A \cap B) = 0.52 + 0.84 - 0.48 = 0.88$
$Pr(\overline{A}) = 1 - Pr(A) = 0.48$ $\qquad Pr(\overline{B}) = 1 - Pr(B) = 0.16$
$Pr(\overline{A} \cup \overline{B}) = Pr(\overline{A \cap B}) = 1 - Pr(A \cap B) = 0.52$
$Pr(\overline{A} \cap \overline{B}) = Pr(\overline{A \cup B}) = 1 - Pr(A \cup B) = 0.12$
$Pr(A \Delta B) = Pr(A) + Pr(B) - 2Pr(A \cap B) = 0.52 + 0.84 - 2(0.48) = 0.4$

11. (a) (i) $\frac{18}{38} + \frac{18}{38} - \frac{9}{38} = \frac{27}{38} \doteq 0.710526$
(ii) $\frac{18}{38} + \frac{18}{38} - \frac{9}{38} = \frac{27}{38} \doteq 0.710526$
(b) (i) $\frac{18}{38} \cdot \frac{18}{38} = \frac{81}{361} \doteq 0.224377$
(ii) $\frac{18}{38} \cdot \frac{2}{38} + \frac{2}{38} \cdot \frac{18}{38} = \frac{9}{361} + \frac{9}{361} = \frac{18}{361} \doteq 0.049861$

12. (i) $Pr(A \cup B) = 1 - Pr(\overline{A \cup B}) = 1 - \frac{1}{5} = \frac{4}{5}$.
(ii) $Pr(A \cup B) = Pr(A) + Pr(B) - Pr(A \cap B)$. Here $Pr(A) = Pr(B)$, so $Pr(A \cup B) + Pr(A \cap B) = 2Pr(A)$, or $2Pr(A) = \frac{4}{5} + \frac{1}{5} = 1$. Hence $Pr(A) = \frac{1}{2}$.
(iii) $Pr(A - B) = P(A \cap \overline{B})$. Since $A = (A \cap B) \cup (A \cap \overline{B})$, where $(A \cap B) \cap (A \cap \overline{B}) = \emptyset$, we have $Pr(A - B) = Pr(A) - Pr(A \cap B) = \frac{1}{2} - \frac{1}{5} = \frac{3}{10}$.
(iv) $Pr(A \Delta B) = Pr(A) + Pr(B) - 2Pr(A \cap B) = \frac{1}{2} + \frac{1}{2} - 2(\frac{1}{5}) = \frac{3}{5}$.

13. $\frac{6}{14} + \frac{6}{14} - \frac{1}{14} = \frac{11}{14}$

14. $[\binom{10}{5}\binom{9}{4} + \binom{10}{6}\binom{9}{3} + \binom{10}{7}\binom{9}{2} + \binom{10}{8}\binom{9}{1}]/[\binom{19}{9} - \binom{9}{9} - \binom{10}{9}]$
$= [(252)(126) + (210)(84) + (120)(36) + (45)(9)]/[92378 - 1 - 10]$
$= (31752 + 17640 + 4320 + 405)/92367 = 54117/92367 \doteq 0.585891$.

15. (a) Ann selects her seven integers in one of $\binom{80}{7}$ ways. Among these possible selections there are $\binom{11}{7}$ that are winning selections. So the probability Ann is a winner is $\binom{11}{7}/\binom{80}{7} = 330/3{,}176{,}716{,}400 \doteq 0.000000104$. [Using a computer algebra system one gets $0.1038808501 \times 10^{-6}$.]
(b) The probability of having two winners is $(0.000000104)^2 \doteq 0.1079123102 \times 10^{-13}$ – **NOT** very likely.

16. In general, $B = B \cap \mathcal{S} = B \cap (A \cup \overline{A}) = (B \cap A) \cup (B \cap \overline{A})$. With $A \subseteq B$ it follows that $B = A \cup (B \cap \overline{A})$, and since $A \cap (B \cap \overline{A}) = B \cap (A \cap \overline{A}) = B \cap \emptyset = \emptyset$, we have $Pr(B) = Pr(A) + Pr(B \cap \overline{A})$. From Axiom (1), $Pr(A), Pr(B \cap \overline{A}) \geq 0$, so $Pr(B) \geq Pr(A)$.

17. Since $A \cup B \subseteq \mathcal{S}$, it follows from the result of the preceding exercise that $Pr(A \cup B) \leq Pr(\mathcal{S}) = 1$. So $1 \geq Pr(A \cup B) = Pr(A) + Pr(B) - Pr(A \cap B)$, and $Pr(A \cap B) \geq Pr(A) + Pr(B) - 1 = 0.7 + 0.5 - 1 = 0.2$.

Section 3.6

1. Let A, B be the events
A: the card drawn is a king
B: the card drawn is an ace or a picture card.
$Pr(A|B) = Pr(A \cap B)/Pr(B) = (\frac{4}{52})/(\frac{16}{52}) = \frac{4}{16} = \frac{1}{4} = 0.25.$

2.

$$\begin{aligned} Pr(A \cap B) &= Pr(A) + Pr(B) - Pr(A \cup B) \\ &= 0.6 + 0.4 - 0.7 = 0.3 \\ Pr(A|B) &= Pr(A \cap B)/Pr(B) = \frac{0.3}{0.4} = \frac{3}{4} = 0.75 \end{aligned}$$

$A = A \cap (B \cup \overline{B}) = (A \cap B) \cup (A \cap \overline{B})$, with $(A \cap B) \cap (A \cap \overline{B}) = A \cap \emptyset = \emptyset$, so $Pr(A) = Pr(A \cap B) + Pr(A \cap \overline{B})$.

Therefore, $Pr(A \cap \overline{B}) = Pr(A) - Pr(A \cap B) = 0.6 - 0.3 = 0.3$, and $Pr(A|\overline{B}) = Pr(A \cap \overline{B})/Pr(\overline{B}) = 0.3/[1 - 0.4] = \frac{0.3}{0.6} = \frac{1}{2} = 0.5$.

3. Let A, B be the events
A: Coach Mollet works his football team throughout August
B: The team finishes as the division champion.
Here $Pr(B|A) = 0.75$ and $Pr(A) = 0.80$, so $Pr(A \cap B) = Pr(A)Pr(B|A) = (0.80)(0.75) = 0.60$.

4. Let A, B be the events
A: a given student is taking calculus
B: a given student is being introduced to a CAS.

(a) Here we want $Pr(B|A)$.
$Pr(A) = (170 + 120)/420 = 29/42$
$Pr(B \cap A) = 170/420 = 17/42$
So $Pr(B|A) = Pr(B \cap A)/Pr(A) = (\frac{17}{42})/(\frac{29}{42}) = \frac{17}{29}$.

(b) In this case the answer is $Pr(\overline{A}|\overline{B})$.
$Pr(\overline{B}) = 1 - Pr(B) = 1 - [(170 + 80)/420] = 1 - \frac{25}{42} = \frac{17}{42}$
$Pr(\overline{A} \cap \overline{B}) = \frac{50}{420} = \frac{5}{42}$
So $Pr(\overline{A}|\overline{B}) = Pr(\overline{A} \cap \overline{B})/Pr(\overline{B}) = (\frac{5}{42})/(\frac{17}{42}) = \frac{5}{17}$.

5. In general, $Pr(A \cup B) = Pr(A) + Pr(B) - Pr(A \cap B)$. Since A, B are independent, $Pr(A \cap B) = Pr(A)Pr(B)$. So

$$\begin{aligned} Pr(A \cup B) &= Pr(A) + Pr(B) - Pr(A)Pr(B) \\ &= Pr(A) + [1 - Pr(A)]Pr(B) \\ &= Pr(A) + Pr(\overline{A})Pr(B). \end{aligned}$$

The proof for $Pr(B) + Pr(\overline{B})Pr(A)$ is similar.

6. Let A, B denote the events

A: first toss is a head

B: three heads are obtained in five tosses.

(a) $Pr(B|A) = Pr(B \cap A)/Pr(A) = \binom{4}{2}(\frac{1}{2})^4/(\frac{1}{2}) = \frac{6}{8} = \frac{3}{4}$. [For the event $B \cap A$ we consider the number of ways we can place two Hs and two Ts in the last four positions. This is $\binom{4}{2}$.]

(b) $Pr(B|\overline{A}) = Pr(B \cap \overline{A})/Pr(\overline{A}) = \binom{4}{3}(\frac{1}{2})^4/(\frac{1}{2}) = \frac{4}{8} = \frac{1}{2}$.

7. Let A, B denote the events

A: Bruno selects a gold coin

B: Madeleine selects a gold coin

$$\begin{aligned}
\text{(a) } Pr(B) &= Pr(B \cap A) + Pr(B \cap \overline{A}) \\
&= Pr(A)Pr(B|A) + Pr(\overline{A})Pr(B|\overline{A}) \\
&= (\tfrac{6}{15})(\tfrac{11}{17}) + (\tfrac{9}{15})(\tfrac{10}{17}) = \tfrac{66+90}{255} = \tfrac{156}{255} = \tfrac{52}{85} \\
\text{(b) } Pr(A|B) &= \tfrac{Pr(A \cap B)}{Pr(B)} = \tfrac{Pr(A)Pr(B|A)}{Pr(B)} \\
&= [(\tfrac{6}{15})(\tfrac{11}{17})]/(\tfrac{52}{85}) = \tfrac{66}{156} = \tfrac{11}{26}.
\end{aligned}$$

8. $A = \{TH, TT\}$, $Pr(A) = (\frac{1}{3})(\frac{2}{3}) + (\frac{1}{3})^2 = \frac{2}{9} + \frac{1}{9} = \frac{3}{9} = \frac{1}{3}$

$B = \{TT, HH\}$, $Pr(B) = (\frac{1}{3})^2 + (\frac{2}{3})^2 = \frac{1}{9} + \frac{4}{9} = \frac{5}{9}$

$A \cap B = \{TT\}$, $Pr(A \cap B) = (\frac{1}{3})^2 = \frac{1}{9}$

$Pr(A \cap B) = \frac{1}{9} = \frac{9}{81} \neq \frac{15}{81} = (\frac{3}{9})(\frac{5}{9}) = Pr(A)Pr(B)$, so A, B are *not* independent.

9.

$$\begin{aligned}
Pr(A \cup B) &= Pr(A) + Pr(B) - Pr(A \cap B) \\
&= Pr(A) + Pr(B) - Pr(A)Pr(B),
\end{aligned}$$

because A, B are independent.

$$\begin{aligned}
0.6 &= 0.3 + Pr(B) - (0.3)Pr(B) \\
0.3 &= 0.7Pr(B)
\end{aligned}$$

So $Pr(B) = \frac{3}{7}$.

10. (a) Let A, B denote the events

A: Alice gets four heads (and three tails)

B: Alice's first toss is a head.

$Pr(A|B) = Pr(A \cap B)/Pr(B) = \frac{[(\frac{1}{2})\binom{6}{3}(\frac{1}{2})^3(\frac{1}{2})^3]}{(\frac{1}{2})} = \binom{6}{3}(\frac{1}{2})^6 = \frac{20}{64} = \frac{5}{16} \doteq 0.3125.$

(b) Let A, C denote the events

A: Alice gets four heads (and three tails)

C: Alice's first and last tosses are heads.

$Pr(A|C) = Pr(A \cap C)/Pr(C) = \frac{[(\frac{1}{2})\binom{5}{2}(\frac{1}{2})^2(\frac{1}{2})^3(\frac{1}{2})]}{(\frac{1}{2})(\frac{1}{2})} = \binom{5}{2}(\frac{1}{2})^5 = \frac{10}{32} = \frac{5}{16} \doteq 0.3125.$

$$\begin{aligned} Pr(A) &= \binom{5}{1}(\tfrac{1}{2})^1(\tfrac{1}{2})^4 + \binom{5}{3}(\tfrac{1}{2})^3(\tfrac{1}{2})^2 + \binom{5}{5}(\tfrac{1}{2})^5 \\ &= (\tfrac{1}{2})^5[5+10+1] = \tfrac{16}{32} = \tfrac{1}{2} \end{aligned}$$

11.
$$\begin{aligned} Pr(B) &= \tfrac{1}{2} \\ Pr(A \cap B) &= (\tfrac{1}{2})[\binom{4}{0}(\tfrac{1}{2})^4 + \binom{4}{2}(\tfrac{1}{2})^2(\tfrac{1}{2})^2 + \binom{4}{4}(\tfrac{1}{2})^4] \\ &= (\tfrac{1}{2})(\tfrac{1}{2})^4[1+6+1] = \tfrac{8}{32} = \tfrac{1}{4} \end{aligned}$$

Since $Pr(A \cap B) = \frac{1}{4} = (\frac{1}{2})(\frac{1}{2}) = Pr(A)Pr(B)$, the events A, B are independent.

12. $(0.95)(0.98) = 0.931$

13. Let A, B be the events

A: Paul initially selects a can of lemonade

B: Betty selects two cans of cola.

$$\begin{aligned} Pr(A|B) &= \tfrac{Pr(A \cap B)}{Pr(B)} = \tfrac{Pr(A)Pr(B|A)}{Pr(B)} \\ Pr(A) &= \tfrac{3}{11} \\ Pr(B) &= Pr(A)Pr(B|A) + Pr(\overline{A})Pr(B|\overline{A}) \\ &= (\tfrac{3}{11})(\tfrac{5}{13})(\tfrac{4}{12}) + (\tfrac{8}{11})(\tfrac{6}{13})(\tfrac{5}{12}) \end{aligned}$$

So $Pr(A|B) = \frac{(\frac{3}{11})(\frac{5}{13})(\frac{4}{12})}{(\frac{3}{11})(\frac{5}{13})(\frac{4}{12}) + (\frac{8}{11})(\frac{6}{13})(\frac{5}{12})} = \frac{60}{60+240} = \frac{6}{30} = \frac{1}{5}$.

14. $Pr(A \cup B \cup C) = Pr(A) + Pr(B) + Pr(C) - Pr(A \cap B) - Pr(A \cap C) - Pr(B \cap C) +$
$Pr(A \cap B \cap C) = Pr(A) + Pr(B) + Pr(C) - Pr(A)Pr(B) - 0 - Pr(B)Pr(C) + 0.$

Note: A, C disjoint $\Rightarrow A \cap C = \emptyset \Rightarrow A \cap B \cap C = \emptyset \Rightarrow Pr(A \cap B \cap C) = 0.$

$$\begin{aligned} 0.8 &= 0.2 + Pr(B) + 0.4 - 0.2Pr(B) - 0.4Pr(B) \\ 0.2 &= 0.4Pr(B) \\ \text{So } Pr(B) &= \tfrac{1}{2} = 0.5 \end{aligned}$$

15. Let A, B denote the events

A: the first component fails

B: the second component fails.

Here $Pr(A) = 0.05$ and $Pr(B|A) = 0.02$. The probability the electronic system fails is $Pr(A \cap B) = Pr(A)Pr(B|A) = (0.05)(0.02) = 0.001$.

16. Let R, B, W denote the withdrawal of a red, blue, and white marble, respectively. Here we are interested in the following cases (with their corresponding probabilities).

RRR:	$Pr(RRR) = (\frac{9}{19})(\frac{8}{18})(\frac{7}{17})$
RRB, RBR, BRR:	$Pr(RRB) = (\frac{9}{19})(\frac{8}{18})(\frac{6}{17})$ $[= Pr(RBR) = Pr(BRR)]$
RRW, RWR, WRR:	$Pr(RRW) = (\frac{9}{19})(\frac{8}{18})(\frac{4}{17})$ $[= Pr(RWR) = Pr(WRR)]$
RBB, BRB, BBR:	$Pr(RBB) = (\frac{9}{19})(\frac{6}{18})(\frac{5}{17})$ $[= Pr(BRB) = Pr(BBR)]$

Consequently, the probability Gayla has withdrawn more red than white marbles is $\frac{(9)(8)(7)+3(9)(8)(6)+3(9)(8)(4)+3(9)(6)(5)}{(19)(18)(17)} = \frac{3474}{5814} = \frac{193}{323} \doteq 0.597523.$

17. In general, $Pr(A \cup B \cup C) = Pr(A) + Pr(B) + Pr(C) - Pr(A \cap B) - Pr(A \cap C) -$
$Pr(B \cap C) + Pr(A \cap B \cap C)$. Since A, B, C are independent we have

$\frac{1}{2} = Pr(A \cup B \cup C) = Pr(A) + Pr(B) + Pr(C) - Pr(A)Pr(B) - Pr(A)Pr(C) - Pr(B)Pr(C) + Pr(A)Pr(B)Pr(C) = (\frac{1}{8}) + (\frac{1}{4}) + Pr(C) - (\frac{1}{8})(\frac{1}{4}) - (\frac{1}{8})Pr(C) - (\frac{1}{4})Pr(C) + (\frac{1}{8})(\frac{1}{4})Pr(C)$.
Consequently, $\frac{1}{2} - \frac{1}{8} - \frac{1}{4} + \frac{1}{32} = [1 - \frac{1}{8} - \frac{1}{4} + \frac{1}{32}]Pr(C)$ and $Pr(C) = \frac{5}{21}$.

18. Let A, B, C, D denote the following events
A: the graphics card comes from the first source
B: the graphics card comes from the second source
C: the graphics card comes from the third source
D: the graphics card is defective.
Then
$Pr(A) = 0.2$, $Pr(B) = 0.35$, $Pr(C) = 0.45$
$Pr(D|A) = 0.05$, $Pr(D|B) = 0.03$, $Pr(D|C) = 0.02$

(a) $Pr(D) = Pr(D \cap A) + Pr(D \cap B) + Pr(D \cap C) = Pr(A)Pr(D|A) + Pr(B)Pr(D|B) + Pr(C)Pr(D|C) = (0.2)(0.05) + (0.35)(0.03) + (0.45)(0.02) = 0.0295$.
So 2.95% of the company's graphics card are defective.
(b) $Pr(C|D) = \frac{Pr(C \cap D)}{Pr(D)} = \frac{Pr(C)Pr(D|C)}{Pr(D)} = [(0.45)(0.02)]/(0.0295) = 18/59 \doteq 0.305085$

19. Here $A = \{HH, HT\}$ and $Pr(A) = \frac{1}{2}$; $B = \{HT, TT\}$ with $Pr(B) = \frac{1}{2}$; and $C = \{HT, TH\}$ with $Pr(C) = \frac{1}{2}$.
Also $A \cap B = \{HT\}$, so $Pr(A \cap B) = \frac{1}{4} = (\frac{1}{2})(\frac{1}{2}) = Pr(A)Pr(B)$; $A \cap C = \{HT\}$, so $Pr(A \cap C) = \frac{1}{4} = (\frac{1}{2})(\frac{1}{2}) = Pr(A)Pr(C)$; and $B \cap C = \{HT\}$ with $Pr(B \cap C) = (\frac{1}{4}) = (\frac{1}{2})(\frac{1}{2}) = Pr(B)Pr(C)$. Consequently, any two of the events A, B, C are independent.
However, $A \cap B \cap C = \{HT\}$ so $Pr(A \cap B \cap C) = \frac{1}{4} \neq \frac{1}{8} = (\frac{1}{2})(\frac{1}{2})(\frac{1}{2}) = Pr(A)Pr(B)Pr(C)$.
Consequently, the events A, B, C are *not* independent.

20. $(0.75)(0.85)(0.9) + (0.75)(0.85)(0.1) + (0.75)(0.15)(0.9) + (0.25)(0.85)(0.9) = 0.57375 + 0.06375 + 0.10125 + 0.19125 = 0.93$.

21. (a) For $0 \leq k \leq 3$, the probability of tossing k heads in three tosses is $\binom{3}{k}(\frac{1}{2})^k(\frac{1}{2})^{3-k} = \binom{3}{k}(\frac{1}{2})^3$. The probability Dustin and Jennifer each toss the same number of heads is $\sum_{k=0}^{3}[\binom{3}{k}\frac{1}{2})^3]^2 = (\frac{1}{2})^6[\binom{3}{0}^2 + \binom{3}{1}^2 + \binom{3}{2}^2 + \binom{3}{3}^2] = (\frac{1}{2})^6[1 + 9 + 9 + 1] = \frac{20}{64} = \frac{5}{16} \doteq 0.3125$.
(b) Let x count the number of heads in Dustin's three tosses and y the number in Jennifer's. Here we consider the cases where $x = 3$: $y = 2, 1$, or 0; $x = 2$: $y = 1$ or 0; $x = 1$: $y = 0$. The probability that Dustin gets more heads than Jennifer is $\binom{3}{3}(\frac{1}{2})^3[\binom{3}{2}(\frac{1}{2})^2(\frac{1}{2}) + \binom{3}{1}(\frac{1}{2})(\frac{1}{2})^2 + \binom{3}{0}(\frac{1}{2})^3] + \binom{3}{2}(\frac{1}{2})^2(\frac{1}{2})[\binom{3}{1}(\frac{1}{2})(\frac{1}{2})^2 + \binom{3}{0}(\frac{1}{2})^3] + \binom{3}{1}(\frac{1}{2})(\frac{1}{2})^2[\binom{3}{0}(\frac{1}{2})^3] = (\frac{1}{2})^6[3+3+1] + (\frac{1}{2})^6(3)[3+1] + (\frac{1}{2})^6(3)(1) = (\frac{1}{2})^6(22) = \frac{11}{32}$.
(c) Here the answer is likewise $\frac{11}{32}$.
[Note: The answers in parts (a), (b), and (c) sum to 1 because the union of the three events for these parts is the entire sample space and the events are disjoint in pairs. Consequently, upon recognizing how the answers in parts (b), (c) are related we see that the answer to part (b) is $(\frac{1}{2})[1 - \frac{5}{16}] = (\frac{1}{2})(\frac{11}{16}) = \frac{11}{32}$.]

22. We need the (equal) probabilities for the disjoint events: (1) One cousin gets a head and the other four get tails; (2) One cousin gets a tail and the other four get heads. The probability for event (1) is $(5)(\frac{1}{2})^5 = \frac{5}{32}$. So the answer is $\frac{5}{32} + \frac{5}{32} = \frac{5}{16}$.

23. Let A, B denote the following events:
A: A new airport-security employee has had prior training in weapon detection
B: A new airport-security employee fails to detect a weapon during the first month on the job.

Here $Pr(A) = 0.9$, $Pr(\overline{A}) = 0.1$, $Pr(B|\overline{A}) = 0.03$ and $Pr(B|A) = 0.005$.

The probability a new airport-security employee, who fails to detect a weapon during the first month on the job, has had prior training in weapon detection $= Pr(A|B) = \frac{Pr(A\cap B)}{Pr(B)} = \frac{Pr(A)Pr(B|A)}{Pr(B\cap A)+Pr(B\cap \overline{A})} = \frac{Pr(A)Pr(B|A)}{Pr(A)Pr(B|A)+Pr(\overline{A})Pr(B|\overline{A})} = (0.9)(0.005)/[0.9)(0.005) + (0.1)(0.03)] = 0.0045/[0.0045 + 0.003] = \frac{45}{75} = \frac{3}{5} = 0.6$.

24. Let A, B, C denote the events
A: the binary string is a palindrome
B: the first and sixth bits of the string are 1
C: the first and sixth bits of the string are the same

(a) $Pr(A|B) = Pr(A \cap B)/Pr(B)$
$Pr(B) = (\frac{1}{2})(1)(1)(1)(1)(\frac{1}{2}) = \frac{1}{4}$, where each 1 is the probability that a given position (second, third, fourth, or fifth) is filled with a 0 or 1.
$Pr(A \cap B) = (\frac{1}{2})(1)(1)(\frac{1}{2})(\frac{1}{2})(\frac{1}{2}) = \frac{1}{16}$, where, for example, the first 1 is the probability that the second position is filled with a 0 or 1, and the third $\frac{1}{2}$ is the probability that the bit in the fifth position matches the bit in the second position.

$$Pr(A|B) = Pr(A \cap B)/Pr(B) = (\frac{1}{16})/(\frac{1}{4}) = \frac{1}{4}$$

(b) $Pr(A|C) = Pr(A \cap C)/Pr(C)$
$Pr(C) = (\frac{1}{2})(1)(1)(1)(1)(\frac{1}{2}) + (\frac{1}{2})(1)(1)(1)(1)(\frac{1}{2})$,
for the two disjoint events where the binary strings start and end with 0, or start and end with 1.
$Pr(A \cap C) = (1)(1)(1)(\frac{1}{2})(\frac{1}{2})(\frac{1}{2}) + (1)(1)(1)(\frac{1}{2})(\frac{1}{2})(\frac{1}{2})$
$Pr(A|C) = [\frac{1}{8} + \frac{1}{8}]/[\frac{1}{4} + \frac{1}{4}] = (\frac{1}{4})/(\frac{1}{2}) = \frac{1}{2}$.

25. (a) There are $\binom{5}{2} = 10$ conditions – one for each pair of events; $\binom{5}{3} = 10$ conditions – one for each triple of events; $\binom{5}{4} = 5$ conditions – one for each quadruple of events; and $\binom{5}{5} = 1$ condition for all five of the events. In total there are 26 $[= 2^5 - \binom{5}{0} - \binom{5}{1}]$ conditions to be checked.
(b) $2^n - \binom{n}{0} - \binom{n}{1} = 2^n - 1 - n = 2^n - (n+1)$ conditions must be checked to establish the independence of n events.

26. Since $0.3 = Pr(\overline{A} \cap \overline{B} = Pr(\overline{A \cup B})$, it follows that $Pr(A \cup B) = 1 - 0.3 = 0.7$.

$Pr(A\Delta B|A\cup B) = \frac{Pr((A\Delta B)\cap(A\cup B))}{Pr(A\cup B)} = Pr(A\Delta B)/Pr(A\cup B) =$
$[Pr(A\cup B) - Pr(A\cap B)]/Pr(A\cup B) = (0.7-0.1)/(0.7) = 0.6/0.7 = 6/7.$

27. Let B_0, B_1, B_2, B_3, and denote A denote the events

B_i: for the three envelopes randomly selected from urn 1 and transferred to urn 2, i envelopes each contain \$1 while the other $3-i$ envelopes each contain \$5, where $0\le i\le 3$.
A: Carmen's selection from urn 2 is an envelope that contains \$1.

Here, $Pr(A) = Pr(A\cap B_0) + Pr(A\cap B_1) + Pr(A\cap B_2) + Pr(A\cap B_3)$
$= Pr(B_0)Pr(A|B_0) + Pr(B_1)Pr(A|B_1) + Pr(B_2)Pr(A|B_2) + Pr(B_3)Pr(A|B_3)$
$= [\binom{6}{0}\binom{8}{3}/\binom{14}{3}](\frac{3}{11}) + [\binom{6}{1}\binom{8}{2}/\binom{14}{3}](\frac{4}{11}) + [\binom{6}{2}\binom{8}{1}/\binom{14}{3}](\frac{5}{11}) + [(\binom{6}{3}\binom{8}{0}/\binom{14}{3}](\frac{6}{11}) =$
$(\frac{2}{13})(\frac{3}{11}) + (\frac{6}{13})(\frac{4}{11}) + (\frac{30}{91})(\frac{5}{11}) + (\frac{5}{91})(\frac{6}{11}) =$
$(\frac{1}{91})(\frac{1}{11})[42+168+150+30] = 390/1001 = 30/77.$

28. $Pr(B|A) < Pr(B) \Rightarrow Pr(B\cap A)/Pr(A) < Pr(B) \Rightarrow Pr(B\cap A) < Pr(A)Pr(B).$

Consequently, $Pr(A\cap B) = Pr(B\cap A) < Pr(A)Pr(B)$, so $Pr(A|B) = Pr(A\cap B)/Pr(B) < Pr(A)$.

29. $0.8 = Pr(A|B) + Pr(B|A) = \frac{Pr(A\cap B)}{Pr(B)} + \frac{Pr(B\cap A)}{Pr(A)} = Pr(A\cap B)[(1/0.3)+(1/0.5)]$, so $(0.15)(0.8) = Pr(A\cap B)[0.5+0.3] = (0.8)Pr(A\cap B)$. Consequently, $Pr(A\cap B) = 0.15$.

30. $Pr(A\cup B) - Pr(A\Delta B) = Pr(A\cap B) = 0.7-0.5 = 0.2$. Since $0.5 = Pr(A|B) = Pr(A\cap B)/Pr(B)$, it follows that $Pr(B) = Pr(A\cap B)/0.5 = 0.2/0.5 = 0.4$.

$0.7 = Pr(A\cup B) = Pr(A) + Pr(B) - Pr(A\cap B) = Pr(A) + 0.4 - 0.2$, so $Pr(A) = 0.5$.

Section 3.7

1. (a) $Pr(X=3) = \frac{1}{4}$
(b) $Pr(X\le 4) = \sum_{x=0}^{4} Pr(X=x) = \frac{1}{8}+\frac{1}{4}+\frac{1}{4}+\frac{1}{4}+\frac{1}{8} = 1$
(c) $Pr(X>0) = \sum_{x=1}^{4} Pr(X=x) = \frac{1}{4}+\frac{1}{4}+\frac{1}{4}+\frac{1}{8} = \frac{7}{8}$
(d) $Pr(1\le X\le 3) = \sum_{x=1}^{3} Pr(X=x) = \frac{1}{4}+\frac{1}{4}+\frac{1}{4} = \frac{3}{4}$
(e) $Pr(X=2|X\le 3) = \dfrac{Pr(X=2 \text{ and } X\le 3)}{Pr(X\le 3)} = \dfrac{Pr(X=2)}{Pr(X\le 3)} = (\dfrac{1}{4})/[\dfrac{1}{8}+\dfrac{1}{4}+\dfrac{1}{4}+\dfrac{1}{4}] =$
$(\dfrac{1}{4})/(\dfrac{7}{8}) = (\dfrac{1}{4})(\dfrac{8}{7}) = \dfrac{2}{7}$
(f) $Pr(X\le 1 \text{ or } X=4) = Pr(X=0) + Pr(X=1) + Pr(X=4) = \frac{1}{8}+\frac{1}{4}+\frac{1}{8} = \frac{1}{2}$

2. (a) $Pr(X=3) = \frac{3(3)+1}{22} = \frac{10}{22} = \frac{5}{11}$
(b) $Pr(X\le 1) = Pr(X=0) + Pr(X=1) = \frac{3(0)+1}{22} + \frac{3(1)+1}{22} = \frac{1}{22} + \frac{4}{22} = \frac{5}{22}$
(c) $Pr(1\le X<3) = Pr(X=1) + Pr(X=2) = \frac{3(1)+1}{22} + \frac{3(2)+1}{22} = \frac{4}{22} + \frac{7}{22} = \frac{11}{22} = \frac{1}{2}$
(d) $Pr(X>-2) = \sum_{x=0}^{3} Pr(X=x) = \frac{3(0)+1}{22} + \frac{3(1)+1}{22} + \frac{3(2)+1}{22} + \frac{3(3)+1}{22} = \frac{1+4+7+10}{22} = \frac{22}{22} = 1$
(e) $Pr(X=1|X\le 2) = \dfrac{Pr(X=1 \text{ and } X\le 2)}{Pr(X\le 2)} = \dfrac{Pr(X=1)}{Pr(X\le 2)} = [\dfrac{3(1)+1}{22}]/[\dfrac{3(0)+1}{22} +$

$\frac{3(1)+1}{22}+\frac{3(2)+1}{22}] = (\frac{4}{22})/(\frac{12}{22}) = \frac{4}{12} = \frac{1}{3}$.

3. (a) $Pr(X=x) = \frac{\binom{10}{x}\binom{110}{5-x}}{\binom{120}{5}}$, $x = 0,1,2,\ldots,5$.

(b) $Pr(X=4) = \frac{\binom{10}{4}\binom{110}{1}}{\binom{120}{5}} = \frac{(210)(110)}{190{,}578{,}024} = \frac{23{,}100}{190{,}578{,}024} = \frac{275}{2{,}268{,}786} \doteq 0.000121$

(c) $Pr(X \geq 4) = Pr(X=4) + Pr(X=5) = \frac{\binom{10}{4}\binom{110}{1}}{\binom{120}{5}} + \frac{\binom{10}{5}\binom{110}{0}}{\binom{120}{5}} = \frac{23{,}100+252}{190{,}578{,}024} = \frac{23{,}352}{190{,}578{,}024} = \frac{139}{1{,}134{,}393} \doteq 0.000123$.

(d) $Pr(X=1|X\leq 2) = \frac{Pr(X=1 \text{ and } X \leq 2)}{Pr(X\leq 2)} = \frac{Pr(X=1)}{Pr(X\leq 2)}$

$= \frac{\binom{10}{1}\binom{110}{4}/\binom{120}{5}}{[(\binom{10}{0}\binom{110}{5}/\binom{120}{5})+(\binom{10}{1}\binom{110}{4}/\binom{120}{5})+(\binom{10}{2}\binom{110}{3}/\binom{120}{5})]}$

$= \binom{10}{1}\binom{110}{4}/[\binom{10}{0}\binom{110}{5} + \binom{10}{1}\binom{110}{4} + \binom{10}{2}\binom{110}{3}]$

$= (10)(5,773,185)/[(1)(122,391,522) + (10)(5,773,185) + (45)(215,820)]$

$= 57,731,850/[122,391,522 + 57,731,850 + 9,711,900]$

$= 57,731,850/189,835,272 = 2675/8796 \doteq 0.304116$.

4. (a)

$$\begin{aligned}
X_1:\quad Pr(X_1=x_1) &= \binom{3}{x_1}(\tfrac{1}{2})^{x_1}(\tfrac{1}{2})^{3-x_1} = \binom{3}{x_1}(\tfrac{1}{2})^3,\ x_1 = 0,1,2,3.\\
X_2:\quad Pr(X_2=x_2) &= \binom{3}{x_2}(\tfrac{1}{2})^{x_2}(\tfrac{1}{2})^{3-x_2} = \binom{3}{x_2}(\tfrac{1}{2})^3,\ x_2 = 0,1,2,3.\\
X\quad Pr(X=-3) &= Pr(X_1=0)Pr(X_2=3|X_1=0) = \binom{3}{0}(\tfrac{1}{2})^3(1) = (\tfrac{1}{2})^3\\
Pr(X=-1) &= Pr(X_1=1)Pr(X_2=2|X_1=1) = \binom{3}{1}(\tfrac{1}{2})^3(1) = (3)(\tfrac{1}{2})^3\\
Pr(X=1) &= Pr(X_1=2)Pr(X_2=1)|X_1=2) = \binom{3}{2}(\tfrac{1}{2})^3(1) = (3)(\tfrac{1}{2})^3\\
Pr(X=3) &= Pr(X_1=3)Pr(X_2=0|X_1=3) = \binom{3}{3}(\tfrac{1}{2})^3(1) = (\tfrac{1}{2})^3
\end{aligned}$$

(b) $E(X_1) = \sum_{x_1=0}^{3} x_1 Pr(X_1=x_1) = 0\cdot\binom{3}{0}(\frac{1}{2})^3 + 1\cdot\binom{3}{1}(\frac{1}{2})^3 + 2\cdot\binom{3}{2}(\frac{1}{2})^3 + 3\cdot\binom{3}{3}(\frac{1}{2})^3 = 0 + \frac{3}{8} + \frac{6}{8} + \frac{3}{8} = \frac{12}{8} = \frac{3}{2}[= 3(\frac{1}{2}) = np$, since X_1 is binomial with $n=3$ and $p=\frac{1}{2}]$.

$E(X_2) = \frac{3}{2}$

$E(X) = (-3)(\frac{1}{2})^3 + (-1)(3)(\frac{1}{2})^3 + (1)(3)(\frac{1}{2})^3 + 3(\frac{1}{2})^3 = 0[= E(X_1) - E(X_2)]$.

5. (a) $Pr(X\geq 3) = \sum_{x=3}^{6} Pr(X=x) = Pr(X=3) + Pr(X=4) + Pr(X=5) + Pr(X=6) = \frac{1}{6}+\frac{1}{6}+\frac{1}{6}+\frac{1}{6} = \frac{4}{6} = \frac{2}{3}$

(b) $Pr(2\leq X\leq 5) = \sum_{x=2}^{5} Pr(X=x) = Pr(X=2)+Pr(X=3)+Pr(X=4)+Pr(X=5) = \frac{1}{6}+\frac{1}{6}+\frac{1}{6}+\frac{1}{6} = \frac{4}{6} = \frac{2}{3}$

(c) $Pr(X=4|X\geq 3) = \frac{Pr(X=4 \text{ and } X\geq 3)}{Pr(X\geq 3)} = \frac{Pr(X=4)}{Pr(X\geq 3)} = (1/6)/(4/6) = 1/4$.

(d) $E(X) = \sum_{x=1}^{6} xPr(X=x) = \sum_{x=1}^{6} x\cdot(\frac{1}{6}) = (\frac{1}{6})(1+2+3+4+5+6) = (\frac{1}{6})(21) = 7/2$.

(e) $E(X^2) = \sum_{x=1}^{6} x^2 Pr(X=x) = (\frac{1}{6})(1+4+9+16+25+36) = (\frac{1}{6})(91) = \frac{91}{6}$

$$\begin{aligned}
\text{Var}\,(X) &= E(X^2) - E(X)^2\\
&= (\tfrac{91}{6}) - (\tfrac{7}{2})^2 = \tfrac{91}{6} - \tfrac{49}{4} = \tfrac{182-147}{12} = \tfrac{35}{12}
\end{aligned}$$

6. (a) $1 = c\sum_{x=1}^{5} \frac{x^2}{x!} = c(1+\frac{4}{2}+\frac{9}{6}+\frac{16}{24}+\frac{25}{120}) = c(\frac{24+48+36+16+5}{24}) = c(\frac{129}{24}) = c(\frac{43}{8})$, so $c = \frac{8}{43}$.

(b) $Pr(X \geq 3) = Pr(X = 3) + Pr(X = 4) + Pr(X = 5) = c(\frac{9}{6} + \frac{16}{24} + \frac{25}{120}) = (\frac{8}{43})(\frac{57}{24}) = \frac{19}{43}$.
(c) $Pr(X = 4|X \geq 3) = \dfrac{Pr(X = 4 \text{ and } X \geq 3)}{Pr(X \geq 3)} = \dfrac{Pr(X = 4)}{Pr(X \geq 3)} = (\dfrac{8}{43})(\dfrac{16}{24})/(\dfrac{19}{43}) = \dfrac{16}{57}$.
(d) $E(X) = \sum_{x=1}^{5} x \cdot Pr(X = x) = (\frac{8}{43}) \sum_{x=1}^{5} x \cdot \frac{x^2}{x!}$
$= (\frac{8}{43}) \sum_{x=1}^{5} \frac{x^3}{x!} = (\frac{8}{43})[1 + \frac{8}{2} + \frac{27}{6} + \frac{64}{24} + \frac{125}{120}]$
$= (\frac{8}{43})(\frac{317}{24}) = \frac{317}{129} \doteq 2.457364$
(e) $E(X^2) = (\frac{8}{43}) \sum_{x=1}^{5} \frac{x^4}{x!} = (\frac{8}{43})[1 + \frac{16}{2} + \frac{81}{6} + \frac{256}{24} + \frac{625}{120}] = (\frac{8}{43})(\frac{921}{24}) = \frac{307}{43}$
$\text{Var }(X) = E(X^2) - E(X)^2 = \frac{307}{43} - (\frac{317}{129})^2 = \frac{18320}{16641} \doteq 1.100895$.

7. (a) $1 = \sum_{x=1}^{5} Pr(X = x) = c \sum_{x=1}^{5} (6 - x) = c(5 + 4 + 3 + 2 + 1) = 15c$, so $c = 1/15$.
(b) $Pr(X \leq 2) = Pr(X = 1) + Pr(X = 2) = (\frac{1}{15})(6 - 1) + (\frac{1}{15})(6 - 2) = \frac{9}{15} = \frac{3}{5}$
(c) $E(X) = \sum_{x=1}^{5} x \cdot Pr(X = x) = \sum_{x=1}^{5} x \cdot (\frac{1}{15})(6 - x)$
$= (\frac{1}{15})[1 \cdot 5 + 2 \cdot 4 + 3 \cdot 3 + 4 \cdot 2 + 5 \cdot 1] = (\frac{1}{15})(35) = \frac{7}{3}$
(d) $E(X^2) = \sum_{x=1}^{5} x^2 \cdot (\frac{1}{15})(6 - x) =$
$(\frac{1}{15})[1 \cdot 5 + 4 \cdot 4 + 9 \cdot 3 + 16 \cdot 2 + 25 \cdot 1] = (\frac{1}{15})(105) = 7$
$\text{Var }(X) = E(X^2) - E(X)^2 = 7 - (\frac{7}{3})^2 = \frac{63-49}{9} = \frac{14}{9}$

8. Let the random variable X count the number of heads in the 100 tosses. Assuming that the tosses are independent, this random variable is binomial with $n = 100$ and $p = \frac{3}{4}$. So Wayne should expect to see $E(X) = np = 100(\frac{3}{4}) = 75$ heads among the results of his 100 tosses.

9. Since X is binomial, $E(X) = 70 = np$ and $\text{Var }(X) = 45.5 = npq$. Hence, we find that $45.5 = 70q$, so $q = 45.5/70 = 0.65$. Consequently, it follows that $p = 0.35$ and $n = 70/p = 70/0.35 = 200$.

10. Let the random variable X denote the player's net winnings and let C denote the cost of playing one round of this carnival game. The probability distribution for X is as follows:

x	$Pr(X = x)$
$5 - C$	$\frac{8}{52} = \frac{2}{13}$
$8 - C$	$\frac{8}{52} = \frac{2}{13}$
$-C$	$\frac{36}{52} = \frac{9}{13}$

Here $0 = E(X) = \frac{2}{13}(5 - C) + \frac{2}{13}(8 - C) + \frac{9}{13}(-C) = \frac{10}{13} + \frac{16}{13} - C$ and $C = \frac{26}{13} = 2$. So the game is fair if the player pays two dollars to play each round.

11. Here X is binomial with $n = 8$ and $p = 0.25$.
(a) $Pr(X = 0) = \binom{8}{0}(0.25)^0(0.75)^8 \doteq 0.100113$
(b) $Pr(X = 3) = \binom{8}{3}(0.25)^3(0.75)^5 \doteq 0.207642$
(c) $Pr(X \geq 6) = Pr(X = 6) + Pr(X = 7) + Pr(X = 8) = \binom{8}{6}(0.25)^6(0.75)^2 + \binom{8}{7}(0.25)^7(0.75)^1 + \binom{8}{8}(0.25)^8(0.75)^0 \doteq 0.004227$.
(d) $Pr(X \geq 6|X \geq 4) = \dfrac{Pr(X \geq 6 \text{ and } X \geq 4)}{Pr(X \geq 4)} = \dfrac{Pr(X \geq 6)}{Pr(X \geq 4)}$

$Pr(X \geq 4) = \sum_{x=4}^{8} \binom{8}{x}(0.25)^x(0.75)^{8-x} = \binom{8}{4}(0.25)^4(0.75)^4 + \binom{8}{5}(0.25)^5(0.75)^3 +$
$\binom{8}{6}(0.25)^6(0.75)^2 + \binom{8}{7}(0.25)^7(0.75)^1 + \binom{8}{8}(0.25)^8(0.75)^0 \doteq 0.113815$
So $Pr(X \geq 6|X \geq 4) \doteq 0.004227/0.113815 \doteq 0.037139$.
(e) $E(X) = np = 8(0.25) = 2$.
(f) $\text{Var}\,(X) = np(1-p) = 8(0.25)(0.75) = 1.5$

12. Here $\sigma_X = \sqrt{9} = 3$.
(a) $Pr(11 \leq X \leq 23) = Pr(11-17 \leq X-17 \leq 23-17) = Pr(-6 \leq X-17 \leq 6)$
$= Pr(|X-17| \leq 6) = Pr(|X-E(X)| \leq 2\sigma_X) \geq 1 = \frac{1}{2^2} = 1 - \frac{1}{4} = \frac{3}{4}$
(b) $Pr(10 \leq X \leq 24) = Pr(|X-17| \leq 7) = Pr(|X-E(X)| \leq (\frac{7}{3})\sigma_X) \geq 1 - 1/(\frac{7}{3})^2 =$
$1 - \frac{9}{49} = \frac{40}{49}$
(c) $Pr(8 \leq X \leq 26) = Pr(|X-17| \leq 9) = Pr(|X-E(X)| \leq 3\sigma_X) \geq 1 - \frac{1}{3^2} = 1 - \frac{1}{9} = \frac{8}{9}$

13. In Chebyshev's Inequality $Pr(|X-E(X)| \leq k\sigma_X) \geq 1 - \frac{1}{k^2}$. If $1 - \frac{1}{k^2} = 0.96$, then $1 - 0.96 = 0.04 = \frac{1}{k^2}$, and $k^2 = \frac{1}{0.04}$. Since $k > 0$ we have $k = \frac{1}{0.2} = 5$.
Here $\text{Var}\,(X) = 4$ so $\sigma_X = 2$ and $c = k\sigma_X = 5 \cdot 2 = 10$.

14. Here X is binomial with $n = 20$ and $p = 1/6$. So $E(X) = np = (20)(\frac{1}{6}) = \frac{20}{6} = \frac{10}{3}$ and $\text{Var}\,(X) = np(1-p) = (20)(\frac{1}{6})(\frac{5}{6}) = \frac{25}{9}$.

15. Let D denote a defective chip and G a good one. Then the sample space $S = \{D, GD, GGD, GGG\}$ and $X(D) = 1$, $X(GD) = 2$, and $X(GGD) = X(GGG) = 3$.
(a)
$$\begin{aligned} Pr(X=1) &= \tfrac{4}{20} = \tfrac{1}{5} \\ Pr(X=2) &= (\tfrac{16}{20})(\tfrac{4}{19}) = \tfrac{16}{95} \\ Pr(X=3) &= (\tfrac{16}{20})(\tfrac{15}{19})(\tfrac{4}{18}) + (\tfrac{16}{20})(\tfrac{15}{19})(\tfrac{14}{18}) = \tfrac{12}{19} \end{aligned}$$
(b) $Pr(X \leq 2) = Pr(X=1) + Pr(X=2) = \frac{1}{5} + \frac{16}{95} = \frac{35}{95} = \frac{7}{19}$
(c) $Pr(X=1|X \leq 2) = \dfrac{Pr(X=1 \text{ and } X \leq 2)}{Pr(X \leq 2)} = \dfrac{Pr(X=1)}{Pr(X \leq 2)} = (\dfrac{1}{5})/(\dfrac{7}{19}) = \dfrac{19}{35}$
(d) $E(X) = \sum_{x=1}^{3} xPr(X=x) = 1(\frac{1}{5}) + 2(\frac{16}{95}) + 3(\frac{12}{19}) = \frac{1}{5} + \frac{32}{95} + \frac{36}{19} = \frac{19+32+180}{95} = \frac{231}{95} =$
2.431579
(e) $E(X^2) = \sum_{x=1}^{3} x^2 Pr(X=x) = 1(\frac{1}{5}) + 4(\frac{16}{95}) + 9(\frac{12}{19}) = \frac{1}{5} + \frac{64}{95} + \frac{108}{19} = \frac{19+64+540}{95} = \frac{623}{95}$

$$\text{Var}\,(X) = E(X^2) - E(X)^2 = \frac{623}{95} - \left(\frac{231}{95}\right)^2 = \frac{5824}{(95)^2} = \frac{5824}{9025} \doteq 0.645319$$

16. (a) $E(aX+b) = \sum_x (ax+b)Pr(X=x) = a\sum_x xPr(X=x) + b\sum_x Pr(X=x) = aE(X)+b$, since $\sum_x Pr(X=x) = 1$.
(b) $\text{Var}\,(aX+b) = \sum_x [(ax+b) - E(aX+b)]^2 Pr(X=x) =$
$\sum_x [(ax+b) - (aE(X)+b)]^2 Pr(X=x)$ [from part (a)] $= \sum_x (ax - aE(X))^2 Pr(X=x) =$
$a^2 \sum_x (x - E(X))^2 Pr(X=x) = a^2\ \text{Var}\,(X)$

17. (a) $E(X(X-1)) = \sum_{x=0}^{n} x(x-1)Pr(X=x)$
$= \sum_{x=2}^{n} x(x-1)Pr(X=x) = \sum_{x=2}^{n} x(x-1)\binom{n}{x}p^x q^{n-x}$
$= \sum_{x=2}^{n} \frac{n!}{x!(n-x)!}x(x-1)p^x q^{n-x}$

$= \sum_{x=2}^{n} \frac{n!}{(x-2)!(n-x)!} p^x q^{n-x} = p^2 n(n-1) \sum_{x=2}^{n} \frac{(n-2)!}{(x-2)!(n-x)!} p^{x-2} q^{n-x}$
$= p^2 n(n-1) \sum_{y=0}^{n-2} \frac{(n-2)!}{y![n-(y+2)]!} p^y q^{n-(y+2)}$, substituting $x - 2 = y$,
$= p^2 n(n-1) \sum_{y=0}^{n-2} \frac{(n-2)!}{y![(n-2)-y]!} p^y q^{(n-2)-y}$
$= p^2 n(n-1)(p+q)^{n-2}$, by the Binomial Theorem
$= p^2 n(n-1)(1)^{n-2} = p^2 n(n-1) = n^2p^2 - np^2$
(b) $\text{Var}(X) = E(X^2) - E(X)^2 = [E(X(X-1)) + E(X)] - E(X)^2 = [(n^2p^2 - np^2) + np] - (np)^2 = n^2p^2 - np^2 + np - n^2p^2 = np - np^2 = np(1-p) = npq.$

18. (a) $Pr(X > 1) = Pr(X = 2) + Pr(X = 3) + Pr(X = 4) = 0.3 + 0.2 + 0.1 = 0.6 = 1 - 0.4 = 1 - Pr(X \leq 1)$
(b) $Pr(X = 3 | X \geq 2) = \dfrac{Pr(X = 3 \text{ and } X \geq 2)}{Pr(X \geq 2)} = Pr(X = 3)/Pr(X \geq 2) =$
$Pr(X = 3)/[Pr(X = 2) + Pr(X = 3) + Pr(X = 4)] = 0.2/0.6 = 1/3$
(c) $E(X) = \sum_{x=1}^{4} x Pr(X = x) = 1(0.4) + 2(0.3) + 3(0.2) + 4(0.1) = 2$
(d) $E(X^2) = \sum_{x=1}^{4} x^2 Pr(X = x) = 1^2(0.4) + 2^2(0.3) + 3^2(0.2) + 4^2(0.1) = 5$
$\text{Var}(X) = E(X^2) - E(X)^2 = 5 - 2^2 = 5 - 4 = 1$

(a)

Word	x, the number of letters and apostrophes in the word
I'll	4
make	4
him	3
an	2
offer	5
he	2
can't	5
refuse	6

19.

x	$Pr(X = x)$
2	$2/8 = 1/4$
3	$1/8$
4	$2/8 = 1/4$
5	$2/8 = 1/4$
6	$1/8$

(b) $E(X) = \sum_{x=2}^{6} x \cdot Pr(X = x)$
$= 2(1/4) + 3(1/8) + 4(1/4) + 5(1/4) + 6(1/8)$
$= (1/8)[4 + 3 + 8 + 10 + 6] = 31/8$

(c) $E(X^2) = \sum_{x=2}^{6} x^2 \cdot Pr(X = x)$
$= 4(1/4) + 9(1/8) + 16(1/4) + 25(1/4) + 36(1/8)$
$= (1/8)[8 + 9 + 32 + 50 + 36] = 135/8$

$\text{Var}(X) = E(X^2) - E(X)^2 = (135/8) - (31/8)^2 = [1080 - 961]/64 = 119/64$

20. (a) $Pr(X = 0) = (0.05)(0.1)(0.12) = 0.0006$
$Pr(X = 1) = (0.95)(0.1)(0.12) + (0.05)(0.9)(0.12) + (0.05)(0.1)(0.88) = 0.0114 + 0.0054 + 0.0044 = 0.0212$

$Pr(X=2) = (0.95)(0.9)(0.12) + (0.95)(0.1)(0.88) + (0.05)(0.9)(0.88) = 0.1026 + 0.0836 + 0.0396 = 0.2258$
$Pr(X=3) = (0.95)(0.9)(0.88) = 0.7524$
[Note that $\sum_{x=0}^{3} Pr(X=x) = 0.0006 + 0.0212 + 0.2258 + 0.7524 = 1.$]
(b) $Pr(X \geq 2 | X \geq 1) = \dfrac{Pr(X \geq 2 \text{ and } X \geq 1)}{Pr((X \geq 1)} = Pr(X \geq 2)/Pr(X \geq 1) = [0.2258 + 0.7524]/[0.0212 + 0.2258 + 0.7524] = 0.9782/0.9994 = 0.978787272$
(c) $E(X) = \sum_{x=0}^{3} x \cdot Pr(X=x) = 0(0.0006) + 1(0.02122) + 2(0.2258) + 3(0.7524) = 2.73.$
(d) $E(X^2) = \sum_{x=0}^{3} x^2 \cdot Pr(X=x) = 0^2(0.0006) + 1^2(0.0212) + 2^2(0.2258) + 3^2(0.7524) = 7.696$
$\text{Var}\ (X) = E(X^2) - E(X)^2 = 7.696 - (2.73)^2 = 0.2431.$

21. $Pr(X=2) = [\binom{1}{1}\binom{1}{1}]/\binom{5}{2} = 1/10$ $\quad Pr(X=3) = [\binom{2}{1}\binom{1}{1}]/\binom{5}{2} = 2/10$
$Pr(X=4) = [\binom{3}{1}\binom{1}{1}]/\binom{5}{2} = 3/10$ $\quad Pr(X=5) = [\binom{4}{1}\binom{1}{1}]/\binom{5}{2} = 4/10$

$E(X) = (1/10)(2) + (2/10)(3) + (3/10)(4) + (4/10)(5) = (1/10)[2+6+12+20] = 40/10 = 4$
$E(X^2) = (1/10)(4) + (2/10)(9) + (3/10)(16) + (4/10)(25) = (1/10)[4 + 18 + 48 + 100] = 170/10 = 17$
$\text{Var}(X) = E(X^2) - E(X)^2 = 17 - 16 = 1$, so $\sigma_X = \sqrt{1} = 1$.

Supplementary Exercises

1. Suppose that $(A-B) \subseteq C$ and $x \in A - C$. Then $x \in A$ but $x \notin C$. If $x \notin B$, then $[x \in A \wedge x \notin B] \Longrightarrow x \in (A-B) \subseteq C$. So now we have $x \notin C$ and $x \in C$. This contradiction gives us $x \in B$, so $(A-C) \subseteq B$.
Conversely, if $(A-C) \subseteq B$, let $y \in A - B$. Then $y \in A$ but $y \notin B$. If $y \notin C$, then $[y \in A \wedge y \notin C] \Longrightarrow y \in (A-C) \subseteq B$. This contradiction, i.e., $y \notin B$ and $y \in B$, yields $y \in C$, so $(A-B) \subseteq C$.

2. Let $S = \{x, y, a_1, a_2, \ldots, a_n\}$. There are $\binom{n+2}{r}$ subsets of S containing r elements, where $r \geq 2$. These subsets fall into three categories. (a) Neither x nor y is in the subset. There are $\binom{n}{r}$ of these. (b) Exactly one of x and y is in the subset. These account for $2\binom{n}{r-1}$ subsets. (c) Both x and y are in the subset. There are $\binom{n}{r-2}$ such subsets.

3. (a) $\mathcal{U} = \{1,2,3\}$, $A = \{1,2\}$, $B = \{1\}$, $C = \{2\}$ provide a counterexample.
(b) $A = A \cap \mathcal{U} = A \cap (C \cup \overline{C}) = (A \cap C) \cup (A \cap \overline{C}) = (A \cap C) \cup (A - C) = (B \cap C) \cup (B - C) = (B \cap C) \cup (B \cap \overline{C}) = B \cap (C \cup \overline{C}) = B \cap \mathcal{U} = B$
(c) The set assignments for part (a) also provide a counterexample for this situation.

4. (a) Consider $m + n$ objects denoted by $\{x_1, x_2, \ldots, x_m\} \cup \{y_1, y_2, \ldots, y_n\}$. Let $A = \{x_1, \ldots, x_m\}$, $B = \{y_1, \ldots, y_n\}$. In selecting r elements from $A \cup B$ we select k elements from A $(0 \leq k \leq m)$, and $(r-k)$ elements from B $(0 \leq r - k \leq n)$. Consequently, the number of subsets of $A \cup B$ with r elements is $\binom{m+n}{r} = \binom{m}{0}\binom{n}{r} + \binom{m}{1}\binom{n}{r-1} + \ldots + \binom{m}{r}\binom{n}{0} =$

$\sum_{k=0}^{r} \binom{m}{k}\binom{n}{r-k}$.

(b) Replace m by n and r by n in part (a), and use the fact that $\binom{n}{k}\binom{n}{n-k} = \binom{n}{k}^2$.

5. (a) 126 (if teams wear different uniforms); 63 (if teams are not distinguishable).
(b) $2^n - 2; (1/2)(2^n - 2)$. $2^n - 2 - 2n; (1/2)(2^n - 2 - 2n)$.

6. (a) False: Let $A = \{0, 1, 2, 3, \ldots\}$, $B = \{0, -1, -2, \ldots\}$. Then A, B are infinite but $|A \cap B| = |\{0\}| = 1$
(b) False: Let $A = \{1, 2\}$ and $B = \mathbf{Z}^+$.
(c) True
(d) False: Let $A = \{1, 2\}$ and $B = \mathbf{Z}^+$.

7. (a) 128 (b) $|A| = 8$

8. (a) 2^7 (b) $\binom{8}{3}(2^7)$ (c) $\binom{8}{3}\binom{7}{5}$

(d)
```
10  Random
20  Dim S(8)
30  For I = 1 To 8
40      S(I) = Int(Rnd * 15) + 1
50      For J = 1 To I - 1
60          If S(I) = S(J) Then GOTO 40
70      Next J
80  Next I
90  C = 0
100 Rem C counts the odd elements of the subset
110 For I = 1 To 8
120     If (S(I)/2) <> Int (S(I)/2) Then C = C + 1
130 Next I
140 Print "The eight-element subset generated ";
150 Print "in this program contains the elements"
160 For I = 1 To 7
170     Print S(I); ",";
180 Next I
190 Print " and "; S(8); ", and "; C;
200 Print " of these elements are odd"
210 End
```

9. Suppose that $(A \cap B) \cup C = A \cap (B \cup C)$ and that $x \in C$. Then $x \in C \Longrightarrow x \in (A \cap B) \cup C \Longrightarrow x \in A \cap (B \cup C) \subseteq A$, so $x \in A$, and $C \subseteq A$.
Conversely, suppose that $C \subseteq A$.

(1) If $y \in (A \cap B) \cup C$, then $y \in A \cap B$ or $y \in C$. (i) $y \in A \cap B \Longrightarrow y \in (A \cap B) \cup (A \cap C) \Longrightarrow y \in A \cap (B \cup C)$. (ii) $y \in C \Longrightarrow y \in A$, since $C \subseteq A$. Also, $y \in C \Longrightarrow y \in B \cup C$. So $y \in A \cap (B \cup C)$. In either case ((i) or (ii)) we have $y \in A \cap (B \cup C)$, so $(A \cap B) \cup C \subseteq A \cap (B \cup C)$.
(2) Now let $z \in A \cap (B \cup C)$. Then $z \in A \cap (B \cup C) = (A \cap B) \cup (A \cap C) \subseteq (A \cap B) \cup C$.
From (1) and (2) it follows that $(A \cap B) \cup C = A \cap (B \cup C)$.

10. (a) Here $|A \cup B| - |A \cap B| = 5$, so there are 2^5 subsets C where $A \cap B \subseteq C \subseteq A \cup B$. The number containing an even number of elements is $\binom{5}{1}$ (for $|C| = 4$) + $\binom{5}{3}$ (for $|C| = 6$) + $\binom{5}{5}$ (for $|C| = 8$) $= 16$.
(b) 2^5; $\binom{5}{0} + \binom{5}{2} + \binom{5}{4} = 16$.

11. (a) $[0, 14/3]$ (b) $\{0\} \cup (6, 12]$ (c) $[0, +\infty)$ (d) $\emptyset$

12. (a) $A \Delta B = (A - B) \cup (B - A) = (B - A) \cup (A - B) = B \Delta A$
(b) $A \Delta \overline{A} = (A - \overline{A}) \cup (\overline{A} - A) = A \cup \overline{A} = \mathcal{U}$
(c) $A \Delta \mathcal{U} = (A - \mathcal{U}) \cup (\mathcal{U} - A) = \emptyset \cup \overline{A} = \overline{A}$
(d) $A \Delta \emptyset = (A - \emptyset) \cup (\emptyset - A) = A \cup \emptyset = A$

13.

(a)

	A	B	$A \cap B$
$\longrightarrow$	0	0	0
$\longrightarrow$	0	1	0
	1	0	0
$\longrightarrow$	1	1	1

Since $A \subseteq B$, we only consider rows 1,2, and 4 of the table. In these rows A and $A \cap B$ have the same column of results, so $A \subseteq B \Longrightarrow A = A \cap B$.

(c)

	A	B	C	$A \cap \overline{C}$	$A \cap \overline{B}$	$B \cap \overline{C}$	$(A \cap \overline{B}) \cup (B \cap \overline{C})$
$\longrightarrow$	0	0	0	0	0	0	0
	0	0	1	0	0	0	0
	0	1	0	0	0	1	1
	0	1	1	0	0	0	0
$\longrightarrow$	1	0	0	1	1	0	1
	1	0	1	0	1	0	1
$\longrightarrow$	1	1	0	1	0	1	1
$\longrightarrow$	1	1	1	0	0	0	0

We consider only rows 1,5,7, and 8. There $A \cap \overline{C} = (A \cap \overline{B}) \cup (B \cap \overline{C})$.

(b) & (d) The results for these parts are derived in a similar manner.

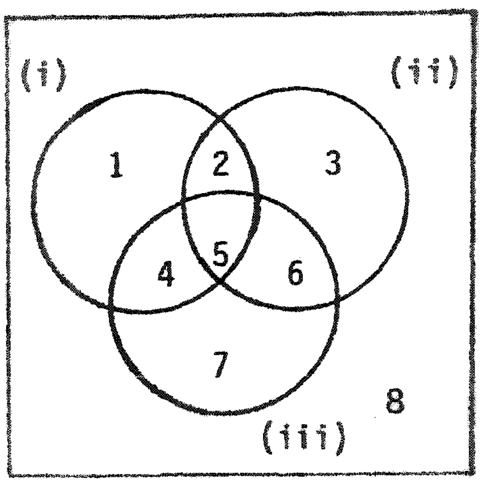
14. (a) $B \subseteq A \Longrightarrow A \cup B = A$.
(b) $(A \cup B = A)$ and $(B \cap C = C) \Longrightarrow A \cap B \cap C = C$.
(c) $C \supseteq B \supseteq A \Longrightarrow (A \cup \overline{B}) \cap (B \cup \overline{C}) = A \cup \overline{C}$.

(d) $(A \cup \overline{B}) \cap (B \cup \overline{A}) = C \Longrightarrow (A \cup \overline{C}) \cap (C \cup \overline{A}) = B$ and $(B \cup \overline{C}) \cap (C \cup \overline{B}) = A$.

15. (a) The r 0's determine $r+1$ locations for the m individual 1's. If $r+1 \geq m$, we can select these locations in $\binom{r+1}{m}$ ways.
(b) Using part (a), here we have k 1's (for the elements of A) and $n-k$ 0's (for the elements in $\mathcal{U} - A$). The $n-k$ 0's provide $n-k+1$ locations for the k 1's so that no two are adjacent. These k locations can be selected in $\binom{n-k+1}{k}$ ways if $n-k+1 \geq k$ or $2k \leq n+1$. So there are $\binom{n-k+1}{k}$ subsets A of $\mathcal{U}$ with $|A| = k$ and such that A contains no consecutive integers.

16. 2/7 **17.** (a) 23 (b) 8 **18.** 7

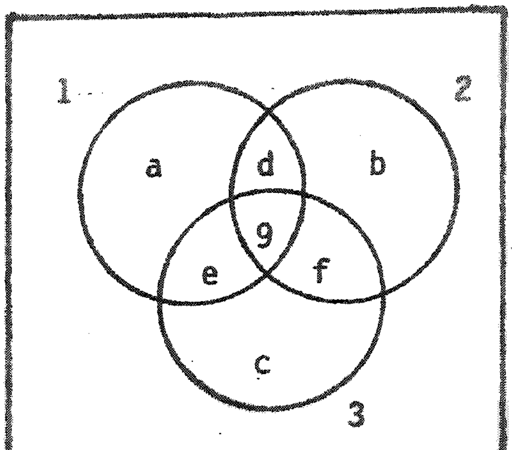
19.



For the given figure let circles (i), (ii), and (iii) denote the subset of assignments where no one is working on experiments 1,2,3, respectively. For each assistant there are seven possibilities: the seven nonempty subsets of $\{1,2,3\}$. So there are 7^{15} possible assignments. To determine the number of assignments in region 8 we need to determine the number of assignments in the union of the three subsets. Region 5 has 0 elements, while regions 2,4,6 each contain 1 element (e.g., for region 2, if all assistants are assigned only to experiment 3 then this is the one way that everyone is working on an experiment, but no one is working on experiments 1 and 2).

In each of regions 1,3,7 there are $3^{15} - 2$ elements (e.g., for regions 1,2,4,5 there are 3 cases to consider where no one is working on experiment 1 – for each assistant can be working on only experiment 2 or only experiment 3 or both experiments 2,3). The number of assignments where at least one person is working on every experiment is $7^{15} - 3[3^{15} - 2] - 3$.

20.



Consider the Venn diagram shown on the left. From the information given we know that
(i) $a+b+c+d+e+f = 21 - 9 = 12$;
(ii) $b+c+f = 5$;
(iii) $a+c+e = 7$; and
(iv) $a+b+d = 6$.
Adding equations (ii), (iii) and (iv) we find that $2(a+b+c) + (d+e+f) = 18$, so $12 = (a+b+c) + [18 - 2(a+b+c)]$, and the number of students who answered exactly one question is $a+b+c = 6$.

21. Since $|A \cap B| = 0$, $|A \cup B| = 12 + 10 = 22$. There are $\binom{22}{7}$ ways to select seven elements from $A \cup B$. Among these selections $\binom{12}{4}\binom{10}{3}$ contain four elements from A and three from B. Consequently, the probability sought here is $\binom{12}{4}\binom{10}{3}/\binom{22}{7} = (495)(120)/(170,544) \doteq 0.3483$.

22. (a) $\mathcal{P}(\mathcal{U}) = \{\emptyset, \{1\}, \{2\}, \{3\}, \{1,2\}, \{1,3\}, \{2,3\}, \mathcal{U}\}$ and $\sum_{A \in \mathcal{P}(\mathcal{U})} \sigma(A) = 1 + 2 + 3 + (1+2) + (1+3) + (2+3) + (1+2+3) = 4(1+2+3) = 2^2(1+2+3) = 24$.
(b) $2^3(1+2+3+4) = 80$
(c) $2^4(1+2+3+4+5) = 240$
(d) $2^{n-1}(1+2+3+\ldots+n)$
Proof (1): Let $x \in \mathcal{U}$. Then x appears in 1 subset by itself, $\binom{n-1}{1}$ subsets of size 2, $\binom{n-1}{2}$ subsets of size 3,..., $\binom{n-1}{k-1}$ subsets of size $k, \ldots,$ and $\binom{n-1}{n-1}$ subsets of size n. Hence x appears in a total of $[\binom{n-1}{0} + \binom{n-1}{1} + \binom{n-1}{2} + \ldots + \binom{n-1}{n-1}] = 2^{n-1}$ subsets. So $\sum_{A \in \mathcal{P}(\mathcal{U})} \sigma(A) = 2^{n-1} \sum_{x \in \mathcal{U}} x = 2^{n-1}(1+2+3+\ldots+n)$.
Proof (2): Let $x \in \mathcal{U}$. For each subset $A \subseteq \mathcal{U}$, if $x \notin A$ then $x \in \overline{A}$. Hence for each pair $(A, \overline{A})$ of subsets of $\mathcal{U}$, exactly one of them contains x. How many such pairs are there? $(1/2)(2^n) = 2^{n-1}$. Consequently, each $x \in \mathcal{U}$ can be found in exactly 2^{n-1} subsets of $\mathcal{U}$ and the result for $\sum_{A \in \mathcal{P}(\mathcal{U})} \sigma(A)$ follows.
(e) Using the result from part (d) it follows that $\sum_{A \in \mathcal{P}(\mathcal{U})} \sigma(A) = 2^{n-1}s$.

23. (a)

$$\underbrace{\binom{16}{8}}_{\text{no diagonal moves}} + \underbrace{\binom{14}{7}}_{\text{one diagonal move}}\binom{15}{1} + \underbrace{\binom{12}{6}}_{\text{two diagonal moves}}\binom{13+2-1}{2}$$

$$+\underbrace{\binom{10}{5}}_{\text{three diagonal moves}}\binom{11+3-1}{3} + \underbrace{\binom{8}{4}}_{\text{four diagonal moves}}\binom{9+4-1}{4} + \underbrace{\binom{6}{3}}_{\text{five diagonal moves}}\binom{7+5-1}{5}$$

$$+\underbrace{\binom{4}{2}}_{\text{six diagonal moves}}\binom{5+6-1}{6} + \underbrace{\binom{2}{1}}_{\text{seven diagonal moves}}\binom{3+7-1}{7} + \underbrace{\binom{0}{0}}_{\text{eight diagonal moves}}\binom{1+8-1}{8}$$

$$= \sum_{i=0}^{8}\binom{2i}{i}\binom{(2i+1)+(8-i)-1}{8-i} = \sum_{i=0}^{8}\binom{2i}{i}\binom{8+i}{8-i}$$

(b) (i) $\binom{12}{6}\binom{14}{2}/\sum_{i=0}^{8}\binom{2i}{i}\binom{8+i}{8-i}$

(ii) $\binom{12}{6}\binom{13}{1}/\sum_{i=0}^{8}\binom{2i}{i}\binom{8+i}{8-i}$

(iii) $[\binom{16}{8}+\binom{12}{6}\binom{14}{2}+\binom{8}{4}\binom{12}{4}+\binom{4}{2}\binom{10}{6}+\binom{0}{0}\binom{8}{8}]/\sum_{i=0}^{8}\binom{2i}{i}\binom{8+i}{8-i}$

24. $x^2-7x=-12 \Rightarrow x^2-7x+12=0 \Rightarrow (x-4)(x-3)=0 \Rightarrow x=4, x=3$.
$x^2-x=6 \Rightarrow x^2-x-6=0 \Rightarrow (x-3)(x+2)=0 \Rightarrow x=3, x=-2$.
Consequently, $A\cap B=\{3\}$ and $A\cup B=\{-2,3,4\}$.

25. $x^2-7x\le -12 \Rightarrow x^2-7x+12\le 0 \Rightarrow (x-3)(x-4)\le 0 \Rightarrow [(x-3)\le 0$ and $(x-4)\ge 0]$ or $[(x-3)\ge 0$ and $(x-4)\le 0] \Rightarrow [x\le 3$ and $x\ge 4]$ or $[x\ge 3$ and $x\le 4] \Rightarrow 3\le x\le 4$, so $A=\{x|3\le x\le 4\}=[3,4]$.

$x^2-x\le 6 \Rightarrow x^2-x-6\le 0 \Rightarrow (x-3)(x+2)\le 0 \Rightarrow [(x-3)\le 0$ and $(x+2)\ge 0]$ or $[(x-3)\ge 0$ and $(x+2)\le 0] \Rightarrow [x\le 3$ and $x\ge -2]$ or $[x\ge 3$ and $x\ge -2] \Rightarrow -2\le x\le 3$, so $B=\{x|-2\le x\le 3\}=[-2,3]$.

Consequently, $A\cap B=\{3\}$ and $A\cup B=[-2,4]$.

26. The probability that all four of these torpedoes fail to destroy the enemy ship is $(1-0.75)(1-0.80)(1-0.85)(1-0.90)=(0.25)(0.20)(0.15)(0.10)=0.00075$. Consequently, the probability the enemy ship is destroyed is $1-0.00075=0.99925$.

27. There are two cases to consider.
(1) The one tail is obtained on the fair coin. The probability for this is $\binom{2}{1}(\frac{1}{2})(\frac{1}{2})\binom{4}{4}(\frac{3}{4})^4$.
(2) The one tail is obtained on the biased coin. The probability in this case is $\binom{2}{2}(\frac{1}{2})^2\binom{4}{3}(\frac{3}{4})^3(\frac{1}{4})$.

Consequently, the answer is the sum of these two probabilities – namely, $\frac{81}{512} + \frac{108}{1024} = \frac{135}{512} \doteq 0.263672$.

28. Let $\mathcal{S}$ be the sample space for an experiment $\mathcal{E}$, with events $A, B \subseteq \mathcal{S}$. Prove that $Pr(A|B) \geq \frac{Pr(A)+Pr(B)-1}{Pr(B)}$.

Proof: $P(A|B) = Pr(A \cap B)/Pr(B) = [Pr(A) + Pr(B) - Pr(A \cup B)]/Pr(B)$. Since $A \cup B \subseteq \mathcal{S}$, it follows that $Pr(A \cup B) \leq Pr(\mathcal{S}) = 1$. Consequently, $-Pr(A \cup B) \geq -1$ and $Pr(A|B) \geq [Pr(A) + Pr(B) - 1]/Pr(B)$.

29. $Pr(A \cap (B \cup C)) = Pr((A \cap B) \cup (A \cap C)) = Pr(A \cap B) + Pr(A \cap C) - Pr((A \cap B) \cap (A \cap C))$. Since A, B, C are independent and $(A \cap B) \cap (A \cap C) = (A \cap A) \cap (B \cap C) = A \cap B \cap C$,

$Pr(A \cap (B \cup C)) = Pr(A)Pr(B) + Pr(A)Pr(C) - Pr(A)Pr(B)Pr(C) = Pr(A)[Pr(B) + Pr(C) - Pr(B)Pr(C)] = Pr(A)[Pr(B) + Pr(C) - Pr(B \cap C] = Pr(A)Pr(B \cup C)$, so A and $B \cup C$ are independent.

30. Suppose we toss a fair coin n times and we let the random variable X count the number of heads among the n tosses. Here we want $Pr(X \geq 2) \geq 0.95$, or $\sum_{k=2}^{n} \binom{n}{k} (\frac{1}{2})^k (\frac{1}{2})^{n-k} = \sum_{k=2}^{n} \binom{n}{k} (\frac{1}{2})^n \geq 0.95$.

Now $\sum_{k=2}^{n} \binom{n}{k} (\frac{1}{2})^n \geq 0.95 \Rightarrow -\sum_{k=2}^{n} \binom{n}{k} (\frac{1}{2})^n \leq -0.95 \Rightarrow 1 - \sum_{k=2}^{n} \binom{n}{k} (\frac{1}{2})^n \leq 1 - 0.95 \Rightarrow \sum_{k=0}^{1} \binom{n}{k} (\frac{1}{2})^n \leq 0.05 \Rightarrow (\frac{1}{2})^n + n(\frac{1}{2})^n = (n+1)(\frac{1}{2})^n \leq 0.05$

For $n = 7$, $(n+1)(\frac{1}{2})^n = 8(\frac{1}{2})^7 = \frac{8}{128} = 0.0625$.
For $n = 8$, $(n+1)(\frac{1}{2})^n = 9(\frac{1}{2})^8 = \frac{9}{256} = 0.035156$
Consequently, the minimum number of tosses is 8.

31. (a) The probability that both tires in any single landing gear blow out is $(0.1)(0.1) = 0.01$. So the probability a landing gear will survive even a hard landing with at least one good tire is $1 - 0.01 = 0.99$.
(b) Assuming the landing gears operate independently of each other, the probability that the jet will be able to land safely even on a hard landing is $(0.99)^3 = 0.970299$.

32. For $A, B \subseteq \mathcal{S}$, $1 = Pr(\mathcal{S}) \geq Pr(A \cup B) = Pr(A) + Pr(B) - Pr(A \cap B)$. Consequently, $Pr(A \cap B) \geq Pr(A) + Pr(B) - 1$.

33. Let A, B denote the events
A: the exit door is open
B: Marlo's selection of two keys includes the one key that opens the exit door

The answer then is given as $Pr(A) + Pr(\overline{A} \cap B) = Pr(A) + Pr(\overline{A})Pr(B) = (\frac{1}{2}) + (\frac{1}{2})[\binom{1}{1}\binom{9}{1}/\binom{10}{2}] = (\frac{1}{2}) + (\frac{1}{2})(\frac{1}{5}) = (\frac{1}{2}) + (\frac{1}{10}) = \frac{6}{10} = \frac{3}{5}$

34. Let A, B, C denote events
A: the first and last outcomes are heads
B: the first and last outcomes are tails

C: the eight tosses result in five heads and three tails.

The answer to the problem is $Pr(C|A \cup B)$. But $Pr(C|A \cup B) = \frac{Pr(C \cap (A \cup B))}{Pr(A \cup B)} = \frac{Pr((C \cap A) \cup (C \cap B))}{Pr(A \cup B)}$.

Since A, B are disjoint, it follows that $C \cap A$, $C \cap B$ are disjoint. Further,
$Pr(A \cup B) = Pr(A) + Pr(B) = (\frac{1}{2})^2 + (\frac{1}{2})^2 = \frac{1}{2}$
$Pr((C \cap A) \cup (C \cap B)) = Pr(C \cap A) + Pr(C \cap B) = (\frac{1}{2})[\binom{6}{3}(\frac{1}{2})^3(\frac{1}{2})^3](\frac{1}{2}) + (\frac{1}{2})[\binom{6}{5}(\frac{1}{2})^5(\frac{1}{2})](\frac{1}{2}) = (\frac{1}{2})^8(20+6) = 13(\frac{1}{2})^7$
Consequently, $Pr(C|A \cup B) = [13(\frac{1}{2})^7]/(\frac{1}{2}) = (13)(\frac{1}{2})^6 = \frac{13}{64}$.

35. $\binom{5}{3}(0.8)^3(0.2)^2 + \binom{5}{4}(0.8)^4(0.2) + \binom{5}{5}(0.8)^5 = 0.2048 + 0.4096 + 0.32768 = 0.94208$

36. $Pr(19,000 \leq X \leq 21,000) = Pr(-1000 \leq X - 20,000 \leq 1000) = Pr(|X - E(X)| \leq 1000)$
Since Var $(X) = 40,000$ boxes^2, we have $\sigma_X = 200$ boxes. So $Pr(|X - E(X)| \leq 1000) = Pr(|X - E(X)| \leq 5\sigma_X) \geq 1 - \frac{1}{5^2} = 1 - \frac{1}{25} = \frac{24}{25} = 0.96$.

37. Success: one head and two tails – the probability for this is $\binom{3}{1}(\frac{1}{2})^1(\frac{1}{2})^2 = \frac{3}{8}$.
$n = 4$, $p = \frac{3}{8}$
Among the four trials (of tossing three fair coins) we want two successes. The probability for this is $\binom{4}{2}(\frac{3}{8})^2(\frac{5}{8})^2 = (6)(9)(25)/2^{12} = 675/2^{11} = \frac{675}{2048}$.

38. (a) $\binom{16}{3}/\binom{22}{3} = \frac{16}{22} \cdot \frac{15}{21} \cdot \frac{14}{20} = \frac{4}{11} \doteq 0.363636$
(b) $\binom{8}{1}\binom{8}{1}\binom{6}{1}/\binom{22}{3} = (3!)(\frac{8}{22})(\frac{8}{21})(\frac{6}{20}) = \frac{96}{385} \doteq 0.249351$
(c) $[\binom{8}{2}\binom{14}{1} + \binom{8}{3}]/\binom{22}{3} = \frac{(3)(8)(7)(14)+(8)(7)(6)}{(22)(21)(20)} = \frac{16}{55} \doteq 0.290909$

39. (a) $1 = \sum_{x=0}^{4} Pr(X = x) = c(0+4) + c(1+4) + c(4+4) + c(9+4) + c(16+4) = c[4+5+8+13+20] = 50c$, so $c = \frac{1}{50} = 0.02$
(b) $Pr(X > 1) = Pr(X \geq 2) = Pr(X = 2) + Pr(X = 3) + Pr(X = 4) = (0.02)(8) + (0.02)(13) + (0.02)(20) = 0.02(8+13+20) = (0.02)(41) = 0.82$
(c) $Pr(X = 3|X \geq 2) = \dfrac{Pr(X = 3 \text{ and } X \geq 2)}{Pr(X \geq 2)} = \dfrac{Pr(X = 3)}{Pr(X \geq 2)} = (0.02)(13)/(0.02)(41) = \dfrac{13}{41} \doteq 0.317073$
(d) $E(X) = \sum_{x=0}^{4} x \cdot Pr(X = x) = 0 \cdot (c)(4) + 1 \cdot (c)(5) + 2 \cdot (c)(8) + 3 \cdot (c)(13) + 4 \cdot (c)(20) = c[5+16+39+80] = 140c = 2.8$
(e) $E(X^2) = \sum_{x=0}^{4} x^2 \cdot Pr(X = x) = 0^2 \cdot (4c) + 1^2 \cdot (5c) + 2^2 \cdot (8c) + 3^2 \cdot (13c) + 4^2 \cdot (20c) = 474c = 9.48$

Var $(X) = E(X^2) - E(X)^2 = 9.48 - (2.8)^2 = 1.64$

40. For each student the probability that all five marbles are green is $(7/11)^5$. Therefore, the probability all 12 students draw only green marbles is $[(7/11)^5]^{12} = (7/11)^{60}$. Consequently, the probability that at least one student draws at least one red marble is $1 - (7/11)^{60}$.

41. (a) To finish with a straight flush, Maureen must draw (i) the 4 and 5 of diamonds; (ii) the 5 and 9 of diamonds; or (iii) the 9 and 10 of diamonds. The probability for each of these three situations is $\binom{2}{2}/\binom{47}{2}$, so the answer is $3/\binom{47}{2}$.
(b) Maureen will finish with a flush if she draws any two of the remaining ten diamonds, which she can do in $\binom{10}{2}$ ways. However, for three choices [as described in part (a)], she actually finishes with a straight flush. Consequently, the answer here is $[\binom{10}{2} - 3]/\binom{47}{2}$.
(c) To finish with a straight from 4 to 8, Maureen must select one of the four 4s and one of the four 5s. This she can do in $\binom{4}{1}\binom{4}{1}$ ways. For the straights from 5 to 9 and 6 to 10 there are likewise $\binom{4}{1}\binom{4}{1}$ possibilities. However, these $3\binom{4}{1}\binom{4}{1}$ straights include three straight flushes, so the answer is $[3\binom{4}{1}\binom{4}{1} - 3]/\binom{47}{2}$.

42. $\binom{2}{1}\binom{2}{1}\binom{2}{1}\binom{2}{1}\binom{2}{1}\binom{2}{1}\binom{2}{1}\binom{2}{1}\binom{32}{4}/\binom{48}{12}$

43. The total number of chips in the grab bag is $1 + 2 + 3 + \cdots + n = n(n+1)/2$, and the probability a chip with i on it is selected is $2i/[n(n+1)]$. Let A, B be the events.
A: the chip with 1 on it is selected
B: a red chip is selected.
$Pr(A|B) = Pr(A \cap B)/Pr(B) = Pr(A)/Pr(B)$.
$Pr(A) = 1/[n(n+1)/2] = 2/[n(n+1)]$.
$Pr(B) = [1+2+3+\cdots+m]/[n(n+1)/2] = [m(m+1)/2]/[n(n+1)/2] = [m(m+1)]/[n(n+1)]$.
Consequently, $Pr(A|B) = \frac{2/[n(n+1)]}{[m(m+1)]/n(n+1)]} = 2/[m(m+1)]$.

44.
(a)

x	$Pr(X = x)$
1	$\binom{6}{1}(1/6)^3 = 1/36$
2	$\binom{6}{2}(1/6)^3 = 15/36 = 5/12$
3	$\binom{6}{3}(1/6)^3 = 20/36 = 5/9$

(b) $E(X) = \sum_{x=1}^{3} x \cdot Pr(X = x) = (1)(1/36) + (2)(15/36) + (3)(20/36)$
$= (1/36)[1 + 30 + 60] = 91/36$

(c) $E(X^2) = \sum_{x=1}^{3} x^2 \cdot Pr(X = x) = (1)(1/36) + (4)(15/36) + (9)(20/36)$
$= (1/36)[1 + 60 + 180] = 241/36$

$\text{Var}(X) = E(X^2) - E(X)^2 = (241/36) - (91/36)^2$
$= [8676 - 8281]/1296 = 395/1296$

45\. (a)

Outcome	Probability of Outcome	x, the number of runs
HHH	$(3/4)^3 = 27/64$	1
HHT	$(3/4)^2(1/4) = 9/64$	2
HTH	$(3/4)(1/4)(3/4) = 9/64$	3
THH	$(1/4)(3/4)^2 = 9/64$	2
HTT	$(3/4)(1/4)^2 = 3/64$	2
THT	$(1/4)(3/4)(1/4) = 3/64$	3
TTH	$(1/4)^2(3/4) = 3/64$	2
TTT	$(1/4)^3 = 1/64$	1

The probability distribution for X:

x	$Pr(X = x)$
1	$(27/64) + (1/64) = 28/64 = 7/16$
2	$(9/64) + (9/64) + (3/64) + (3/64) = 24/64 = 3/8$
3	$(9/64) + (3/64) = 12/64 = 3/16$

(b) $E(X) = \sum_{x=1}^{3} x \cdot Pr(X = x) = (1)(7/16) + (2)(3/8) + (3)(3/16)$
$= (1/16)[7 + 12 + 9] = 28/16 = 7/4$

(c) $E(X^2) = \sum_{x=1}^{3} x^2 \cdot Pr(X = x) = (1)(7/16) + (4)(3/8) + (9)(3/16)$
$= (1/16)[7 + 24 + 27] = 58/16 = 29/8$

Var $(X) = E(X^2) - E(X)^2 = (29/8) - (7/4)^2 = (29/8) - (49/16)$
$= (58 - 49)/16 = 9/16$

So $\sigma_X = \sqrt{9/16} = 3/4.$

CHAPTER 4
PROPERTIES OF THE INTEGERS: MATHEMATICAL INDUCTION

Section 4.1

1. (a) $S(n): 1^2 + 3^2 + 5^2 + \ldots + (2n-1)^2 = (n)(2n-1)(2n+1)/3$.
$S(1): 1^2 = (1)(1)(3)/3$. This is true.
Assume $S(k): 1^2 + 3^2 + \ldots + (2k-1)^2 = (k)(2k-1)(2k+1)/3$, for some $k \geq 1$.
Consider $S(k+1)$. $[1^2+3^2+\ldots+(2k-1)^2]+(2k+1)^2 = [(k)(2k-1)(2k+1)/3]+(2k+1)^2 = [(2k+1)/3][k(2k-1)+3(2k+1)] = [(2k+1)/3][2k^2+5k+3] = (k+1)(2k+1)(2k+3)/3$, so $S(k) \Longrightarrow S(k+1)$ and the result follows for all $n \in \mathbf{Z}^+$ by the Principle of Mathematical Induction.
(c) $S(n): \sum_{i=1}^{n} \frac{1}{i(i+1)} = \frac{n}{n+1}$
$S(1): \sum_{i=1}^{1} \frac{1}{i(i+1)} = \frac{1}{1(2)} = \frac{1}{1+1}$, so $S(1)$ is true.
Assume $S(k): \sum_{i=1}^{k} \frac{1}{i(i+1)} = \frac{k}{k+1}$. Consider $S(k+1)$.
$\sum_{i=1}^{k+1} \frac{1}{i(i+1)} = \sum_{i=1}^{k} \frac{1}{i(i+1)} + \frac{1}{(k+1)(k+2)} = \frac{k}{(k+1)} + \frac{1}{(k+1)(k+2)} = [k(k+2)+1]/[(k+1)(k+2)] = (k+1)/(k+2)$, so $S(k) \Longrightarrow S(k+1)$ and the result follows for all $n \in \mathbf{Z}^+$ by the Principle of Mathematical Induction.

The proofs of the remaining parts are similar.

2. (a) $S(n): \sum_{i=1}^{n} 2^{i-1} = 2^n - 1$
$S(1): \sum_{i=1}^{1} 2^{i-1} = 2^{1-1} = 2^1 - 1$, so $S(1)$ is true.
Assume $S(k): \sum_{i=1}^{k} 2^{i-1} = 2^k - 1$. Consider $S(k+1)$.
$\sum_{i=1}^{k+1} 2^{i-1} = \sum_{i=1}^{k} 2^{i-1} + 2^k = 2^k - 1 + 2^k = 2^{k+1} - 1$, so $S(k) \Longrightarrow S(k+1)$ and the result is true for all $n \in \mathbf{Z}^+$ by the Principle of Mathematical Induction.
(b) For $n = 1, \sum_{i=1}^{1} i(2^i) = 2 = 2 + (1-1)2^{1+1}$, so the statement $S(1)$ is true. Assume $S(k)$ true – that is, $\sum_{i=1}^{k} i(2^i) = 2 + (k-1)2^{k+1}$. For $n = k+1, \sum_{i=1}^{k+1} i(2^i) = \sum_{i=1}^{k} i(2^i) + (k+1)2^{k+1} = 2 + (k-1)2^{k+1} + (k+1)2^{k+1} = 2 + (2k)2^{k+1} = 2 + k \cdot 2^{k+2}$, so $S(n)$ is true for all $n \in \mathbf{Z}^+$ by the Principle of Mathematical Induction.
(c) For $n = 1$, we find that $\sum_{i=1}^{1}(i)(i!) = 1 = (1+1)! - 1$, so $S(1)$ is true. We assume the truth of $S(k)$ – that is, $\sum_{i=1}^{k} i(i!) = (k+1)! - 1$. Now for the case where $n = k+1$ we have $\sum_{i=1}^{k+1} i(i!) = \sum_{i=1}^{k} i(i!)+(k+1)(k+1)! = (k+1)!-1+(k+1)(k+1)! = [1+(k+1)](k+1)!-1 = (k+2)(k+1)!-1 = (k+2)!-1$. Hence $S(k) \Longrightarrow S(k+1)$, and since $S(1)$ is true it follows that the statement is true for all $n \geq 1$, by the Principle of Mathematical Induction.

3. (a) From $\sum_{i=1}^{n} i^3 + (n+1)^3 = \sum_{i=0}^{n}(i^3+3i^2+3i+1) = \sum_{i=1}^{n} i^3 + 3\sum_{i=1}^{n} i^2 + 3\sum_{i=1}^{n} i + \sum_{i=0}^{n} 1$, we have $(n+1)^3 = 3\sum_{i=1}^{n} i^2 + 3\sum_{i=1} i + (n+1)$. Consequently,

$$\begin{aligned} 3\sum_{i=1}^{n} i^2 &= (n^3+3n^2+3n+1) - 3[(n)(n+1)/2] - n - 1 \\ &= n^3 + (3/2)n^2 + (1/2)n \\ &= (1/2)[2n^3+3n^2+n] = (1/2)n(2n^2+3n+1) \\ &= (1/2)n(n+1)(2n+1), \text{ so} \end{aligned}$$

$\sum_{i=1}^{n} i^2 = (1/6)n(n+1)(2n+1)$ (as shown in Example 4.4).
(b) From $\sum_{i=1}^{n} i^4 + (n+1)^4 = \sum_{i=0}^{n}(i+1)^4 = \sum_{i=0}^{n}(i^4+4i^3+6i^2+4i+1) = \sum_{i=1}^{n} i^4 + 4\sum_{i=1}^{n} i^3 + 6\sum_{i=1}^{n} i^2 + 4\sum_{i=1}^{n} i + \sum_{i=0}^{n} 1$, it follows that $(n+1)^4 = 4\sum_{i=1}^{n} i^3 + 6\sum_{i=1}^{n} i^2 + 4\sum_{i=1}^{n} i + \sum_{i=0}^{n} 1$. Consequently,
$4\sum_{i=1}^{n} i^3 = (n+1)^4 - 6[n(n+1)(2n+1)/6] - 4[n(n+1)/2] - (n+1) = n^4+4n^3+6n^2+4n+1-(2n^3+3n^2+n)-(2n^2+2n)-(n+1) = n^4+2n^3+n^2 = n^2(n^2+2n+1) = n^2(n+1)^2$.
So $\sum_{i=1}^{n} i^3 = (1/4)n^2(n+1)^2$ [as shown in part (d) of Exercise 1 for this section].

From $\sum_{i=1}^{n} i^5 + (n+1)^5 = \sum_{i=0}^{n}(i+1)^5 = \sum_{i=0}^{n}(i^5+5i^4+10i^3+10i^2+5i+1) = \sum_{i=1}^{n} i^5 + 5\sum_{i=1}^{n} i^4 + 10\sum_{i=1}^{n} i^3 + 10\sum_{i=1}^{n} i^2 + 5\sum_{i=1}^{n} i + \sum_{i=0}^{n} 1$, we have $5\sum_{i=1}^{n} i^4 = (n+1)^5 - (10/4)n^2(n+1)^2 - (10/6)n(n+1)(2n+1) - (5/2)n(n+1) - (n+1)$. So

$$\begin{aligned} 5\sum_{i=1}^{n} i^4 &= n^5+5n^4+10n^3+10n^2+5n+1-(5/2)n^4 \\ &\quad -5n^3-(5/2)n^2-(10/3)n^3-5n^2-(5/3)n-(5/2)n^2-(5/2)n-n-1 \\ &= n^5+(5/2)n^4+(5/3)n^3-(1/6)n. \end{aligned}$$

Consequently, $\sum_{i=1}^{n} i^4 = (1/30)n(n+1)(6n^3+9n^2+n-1)$.

4. Let $x_1, x_2, \ldots, x_{25}$ denote the numbers (in their order on the wheel), and assume that $x_1+x_2+x_3 < 39, x_2+x_3+x_4 < 39, \ldots, x_{24}+x_{25}+x_1 < 39$, and $x_{25}+x_1+x_2 < 39$. Then $\sum_{i=1}^{25} 3x_i < 25(39)$. But $\sum_{i=1}^{25} 3x_i = 3\sum_{i=1}^{25} i = (3)(25)(26)/2 = (39)(25)$.

5. (a) 7626 (b) 627,874

6. a) The typical palindrome under study here has the form *abba* where $1 \le a \le 9$ and $0 \le b \le 9$. Consequently there are $9 \cdot 10 = 90$ such palindromes, by the rule of product. Their sum is $\sum_{a=1}^{9}(\sum_{b=0}^{9} abba) = \sum_{a=1}^{9}\sum_{b=0}^{9}(1001a+110b) = \sum_{a=1}^{9}[10(1001a)+110\sum_{b=0}^{9} b] = \sum_{a=1}^{9}(10010a+110(9\cdot 10/2)) = 10010\sum_{a=1}^{9} a + \sum_{a=1}^{9} 4950 = 10010(9\cdot 10/2)+9(4950) = 450450+44550 = 495000$.

b)
```
begin
    sum := 0
    for a := 1 to 9 do
        for b := 0 to 9 do
            sum := sum + 1001 * a + 110 * b
    print sum
end
```

7.

$$\begin{aligned}
4n+110 &= 6+8+10+\cdots+[6+(n-1)2] \\
&= 6n+[0+2+4+\cdots+(n-1)2] \\
&= 6n+2[1+2+\cdots+(n-1)] \\
&= 6n+2[(n-1)(n)/2] \\
&= 6n+(n-1)(n)=n^2+5n \\
n^2+n-110 &= (n+11)(n-10)=0,
\end{aligned}$$

so $n=10$ – the number of layers.

8. Here we have $\sum_{i=1}^{n} i^2=(n)(n+1)(2n+1)/6=(2n)(2n+1)/2=\sum_{i=1}^{2n} i$,

and $(n)(n+1)(2n+1)/6=(2n)(2n+1)/2 \Rightarrow (n)(n+1)/6=(2n)/2 \Rightarrow$

$(n+1)/6=1 \Rightarrow n+1=6 \Rightarrow n=5$.

9. (a) $\sum_{i=11}^{33} i=\sum_{i=1}^{33} i-\sum_{i=1}^{10} i=[(33)(34)/2]-[(10)(11)/2]=561-55=506$

(b) $\sum_{i=11}^{33} i^2=\sum_{i=1}^{33} i^2-\sum_{i=1}^{10} i^2=[(33)(34)(67)/6]-[(10)(11)(21)/6]=12144$

10. $\sum_{i=51}^{100} t_i=\sum_{i=1}^{100} t_i-\sum_{i=1}^{50} t_i=(100)(101)(102)/6-(50)(51)(52)/6=171,700-22,100=149,600$.

11. a) $\sum_{i=1}^{n} t_{2i}=\sum_{i=1}^{n} \frac{(2i)(2i+1)}{2}=\sum_{i=1}^{n}(2i^2+i)=2\sum_{i=1}^{n} i^2+\sum_{i=1}^{n} i=2[(n)(n+1)(2n+1)/6]+[n(n+1)/2]=[n(n+1)(2n+1)/3]+[n(n+1)/2]=n(n+1)[\frac{2n+1}{3}+\frac{1}{2}]=n(n+1)[\frac{4n+5}{6}]=n(n+1)(4n+5)/6$.

b) $\sum_{i=1}^{100} t_{2i}=100(101)(405)/6=681,750$.

c)
```
begin
    sum := 0
    for i := 1 to 100 do
        sum := sum + (2 * i) * (2 * i + 1)/2
    print sum
end
```

12. (a) $(\cos\theta+i\sin\theta)^2=\cos^2\theta+2i\sin\theta\cos\theta-\sin^2\theta=(\cos^2\theta-\sin^2\theta)+i(2\sin\theta\cos\theta)=\cos 2\theta+i\sin 2\theta$.

(b) $S(n):(\cos\theta+i\sin\theta)^n=\cos n\theta+i\sin n\theta$. $S(1)$ is true, so assume $S(k):(\cos\theta+i\sin\theta)^k=(\cos k\theta+i\sin k\theta)$. Consider $S(k+1):(\cos\theta+i\sin\theta)^{k+1}=(\cos\theta+i\sin\theta)^k(\cos\theta+i\sin\theta)=(\cos k\theta+i\sin k\theta)\cdot(\cos\theta+i\sin\theta)=(\cos k\theta\cos\theta-\sin k\theta\sin\theta)+i(\sin\theta\cos k\theta+\sin k\theta\cos\theta)=\cos(k+1)\theta+i\sin(k+1)\theta$. So $S(k)\Longrightarrow S(k+1)$ and the result is true for all $n\in\mathbf{Z}^+$ by the Principle of Mathematical Induction.

(c) $(1+i)^{100}=2^{50}(\cos 4500^\circ+i\sin 4500^\circ)=2^{50}(\cos 180^\circ+i\sin 180^\circ)=-(2^{50})$.

13. (a) There are $49(= 7^2)$ 2×2 squares and $36(= 6^2)$ 3×3 squares. In total there are $1^2 + 2^2 + 3^2 + \cdots + 8^2 = (8)(8+1)(2 \cdot 8 + 1)/6 = (8)(9)(17)/6 = 204$ squares.

(b) For each $1 \leq k \leq n$ the $n \times n$ chessboard contains $(n-k+1)^2$ $k \times k$ squares. In total there are $1^2 + 2^2 + 3^2 + \cdots + n^2 = n(n+1)(2n+1)/6$ squares.

14. For $n = 4$ we have $2^4 = 16 < 24 = 4!$, so the statement $S(4)$ is true. Assume the truth of $S(k)$ – that is, $2^k < k!$. For $k \geq 4, 2 < k+1$, and $[(2^k < k!) \wedge (2 < k+1)] \Longrightarrow (2^k)(2) < (k!)(k+1)$, or $2^{k+1} < (k+1)!$ Hence $S(n)$ is true for all $n \geq 4$ by the Principle of Mathematical Induction.

15. For $n = 5, 2^5 = 32 > 25 = 5^2$. Assume the result for $n = k(\geq 5) : 2^k > k^2$. For $k > 2, k(k-2) > 1$, or $k^2 > 2k+1$. But $2^k > k^2 \Longrightarrow 2^k + 2^k > k^2 + k^2 \Longrightarrow 2^{k+1} > k^2 + k^2 > k^2 + (2k+1) = (k+1)^2$. Hence the result is true for $n \geq 5$ by the Principle of Mathematical Induction.

16. (a) 3 (b) $s_2 = 2$; $s_4 = 4$

(c) For $n \geq 1$, $s_n = \sum_{\emptyset \neq A \subseteq X_n} \frac{1}{p_A} = n$.

Proof: For $n = 1$, $s_1 = \frac{1}{1} = 1$, so this first case is true and establishes the basis step. Now, for the inductive step, assume the result true for $n = k(\geq 1)$. That is, $s_k = \sum_{\emptyset \neq A \subseteq X_k} \frac{1}{p_A} = k$. For $n = k+1$ we find that $s_{k+1} = \sum_{\emptyset \neq A \subseteq X_{k+1}} \frac{1}{p_A} = \sum_{\emptyset \neq B \subseteq X_k} \frac{1}{p_B} + \sum_{\{k+1\} \subseteq C \subseteq X_{k+1}} \frac{1}{p_C}$, where the first sum is taken over all nonempty subsets B of X_k and the second sum over all subsets C of X_{k+1} that contain $k+1$. Then $s_{k+1} = s_k + [(\frac{1}{k+1}) + (\frac{1}{k+1})s_k] = k + (\frac{1}{k+1}) + (\frac{1}{k+1})k = k + (\frac{1}{k+1})(1+k) = k+1$. Consequently, we have deduced the truth for $n = k+1$ from that of $n = k$. The result now follows for all $n \geq 1$ by the Principle of Mathematical Induction.

17. (a) Once again we start at $n = 0$. Here we find that $1 = 1 + (0/2) \leq H_1 = H_{2^0}$, so this first case is true. Assuming the truth for $n = k(\in \mathbf{N})$ we obtain the induction hypothesis

$$1 + (k/2) \leq H_{2^k}.$$

Turning now to the case where $n = k+1$ we find $H_{2^{k+1}} = H_{2^k} + [1/(2^k+1)] + [1/(2^k+2)] + \ldots + [1/(2^k+2^k)] \geq H_{2^k} + [1/(2^k+2^k)] + [1/(2^k+2^k)] + \ldots + [1/(2^k+2^k)] = H_{2^k} + 2^k[1/2^{k+1}] = H_{2^k} + (1/2) \geq 1 + (k/2) + (1/2) = 1 + (k+1)/2$.

The result now follows for all $n \geq 0$ by the Principle of Mathematical Induction.

(b) Starting with $n = 1$ we find that

$$\sum_{j=1}^{1} jH_j = H_1 = 1 = [(2)(1)/2](3/2) - [(2)(1)/4] = [(2)(1)/2]H_2 - [(2)(1)/4].$$

Assuming the truth of the given statement for $n = k$, we have

$$\sum_{j=1}^{k} jH_j = [(k+1)(k)/2]H_{k+1} - [(k+1)(k)/4].$$

For $n = k+1$ we now find that

$$\sum_{j=1}^{k+1} jH_j = \sum_{j=1}^{k} jH_j + (k+1)H_{k+1}$$

$$= [(k+1)(k)/2]H_{k+1} - [(k+1)(k)/4] + (k+1)H_{k+1}$$

$$= (k+1)[1+(k/2)]H_{k+1} - [(k+1)(k)/4]$$

$$= (k+1)[1+(k/2)][H_{k+2} - (1/(k+2))] - [(k+1)(k)/4]$$

$$= [(k+2)(k+1)/2]H_{k+2} - [(k+1)(k+2)]/[2(k+2)] - [(k+1)(k)/4]$$

$$= [(k+2)(k+1)/2]H_{k+2} - [(1/4)[2(k+1) + k(k+1)]]$$

$$= [(k+2)(k+1)/2]H_{k+2} - [(k+2)(k+1)/4].$$

Consequently, by the Principle of Mathematical Induction, it follows that the given statement is true for all $n \in \mathbf{Z}^+$.

18. Conjecture: For all $n \in \mathbf{N}$, $(n^2+1)+(n^2+2)+(n^2+3)+\ldots+(n+1)^2 = \sum_{i=1}^{2n+1}(n^2+i) = n^3+(n+1)^3$.
Proof: $\sum_{i=1}^{2n+1}(n^2+i) = n^2\sum_{i=1}^{2n+1}1 + \sum_{i=1}^{2n+1}i = n^2(2n+1)+(2n+1)(2n+2)/2 = 2n^3+n^2+(2n+1)(n+1) = 2n^3+n^2+2n^2+3n+1 = n^3+[n^3+3n^2+3n+1] = n^3+(n+1)^3$.

19. Assume $S(k)$ true for some $k \geq 1$. For $S(k+1), \sum_{i=1}^{k+1} i = [k+(1/2)]^2/2+(k+1) = ((k^2+k)+(1/4)+2k+2)/2 = [(k+1)^2+(k+1)+(1/4)]/2 = [(k+1)+(1/2)]^2/2$.
S $S(k) \Longrightarrow S(k+1)$. However, we have no first value of k where $S(k)$ is true:
For each $k \geq 1, \sum_{i=1}^{k} i = (k)(k+1)/2$ and $(k)(k+1)/2 = [k+(1/2)]^2/2 \Longrightarrow 0 = 1/4$.

20. For $n = 0, S = \{a_1\}$ and 0 comparisons are required. Since $0 = 0 \cdot 2^0$, the result is true when $n = 0$. Assume the result for $n = k(\geq 0)$ and consider the case $n = k+1$. If $|S| = 2^{k+1}$ then $S = S_1 \cup S_2$ where $|S_1| = |S_2| = 2^k$. By the induction hypothesis the number of comparisons needed to place the elements in each of S_1, S_2 in ascending order is bounded by $k \cdot 2^k$. Therefore, by the given information, the elements in S can be placed in ascending order by making at most a total of $(k\cdot 2^k)+(k\cdot 2^k)+(2^k+2^k-1) = (k+1)2^{k+1}-1 \leq (k+1)2^{k+1}$ comparisons.

21. For $x, n \in \mathbf{Z}^+$, let $S(n)$ denote the statement: If the program reaches the top of the **while** loop, after the two loop instructions are executed $n(> 0)$ times, then the value of the integer variable *answer* is $x(n!)$.
First consider $S(1)$, the statement for the case where $n = 1$. Here the program (if it reaches the top of the **while** loop) will result in one execution of the **while** loop: x will be assigned the value $x \cdot 1 = x(1!)$, and the value of n will be decreased to 0. With the value of n equal to 0 the loop is not processed again and the value of the variable *answer* is $x(1!)$. Hence $S(1)$ is true.
Now assume the truth for $n = k$: For $x, k \in \mathbf{Z}^+$, if the program reaches the top of the **while** loop, then upon exiting the loop, the value of the variable *answer* is $x(k!)$. To establish $S(k+1)$, if the program reaches the top of the **while** loop, then the following occur during the first execution:

The value assigned to the variable x is $x(k+1)$.
The value of n is decreased to $(k+1) - 1 = k$.

But then we can apply the induction hypothesis to the integers $x(k+1)$ and k, and after we exit the **while** loop for these values, the value of the variable *answer* is $(x(k+1))(k!) = x(k+1)!$
Consequently, $S(n)$ is true for all $n \geq 1$, and we have verified the correctness of this program segment by using the Principle of Mathematical Induction.

22. If $n = 0$, then the statement '$n \neq 0$' is false so the **while** loop is bypassed and the value assigned to *answer* is $x = x + 0 \cdot y$. So the result is true in the first case.
Now assume the result true for $n = k$ – that is, for $x, y \in \mathbf{R}$, if the program reaches the top of the **while** loop with $k \in \mathbf{Z}$, $k \geq 0$, then upon bypassing the loop when $k = 0$, or executing the two loop instructions $k(> 0)$ times, then the value assigned to *answer* is $x + ny$. To establish the result for $n = k + 1$, suppose the program reaches the top of the **while** loop. Since $k \geq 0, n = k + 1 > 0$, so the loop is not bypassed. During the first pass through the **while** loop we find that

The value assigned to x is $x + y$; and
The value of n is decreased to $(k+1) - 1 = k$.

Now we apply the induction hypothesis to the real numbers $x+y$ and y and the nonnegative integer $n - 1 = k$, and upon bypassing the loop when $k = 0$, or executing the two loop instructions $k(> 0)$ times, then the value assigned to *answer* is

$$(x + y) + ky = x + (k+1)y.$$

The result now follows for all $n \in \mathbf{N}$ by the Principle of Mathematical Induction.

23. (a) The result is true for $n = 2, 4, 5, 6$. Assume the result is true for all $n = 2, 4, 5, \ldots, k-1, k$, where $k \geq 6$. If $n = k + 1$, then $n = 2 + (k - 1)$, and since the result is true for $k - 1$, it follows by induction that it is true for $k + 1$. Consequently, by the Alternative Form of the Principle of Mathematical Induction, every $n \in \mathbf{Z}^+, n \neq 1, 3$, can be written as a sum of 2's and 5's.

(b) $24 = 5 + 5 + 7 + 7$ $25 = 5 + 5 + 5 + 5 + 5$ $26 = 5 + 7 + 7 + 7$

$27 = 5+5+5+5+7$ $\qquad$ $28 = 7+7+7+7$

Hence the result is true for all $24 \leq n \leq 28$. Assume the result true for $24 \leq n \leq 28 \leq k$, and consider $n = k+1$. Since $k+1 \geq 29$, we may write $k+1 = [(k+1)-5]+5 = (k-4)+5$, where $k-4$ can be expressed as a sum of 5's and 7's. Hence $k+1$ can be expressed as such a sum and the result follows for all $n \geq 24$ by the Alternative Form of the Principle of Mathematical Induction.

24. (a) $a_3 = 3 \quad a_4 = 5 \quad a_5 = 8 \quad a_6 = 13 \quad a_7 = 21$

(b) $a_1 = 1 < (7/4)^1$, so the result is true for $n = 1$. Likewise, $a_2 = 2 < \frac{49}{16} = (7/4)^2$ and the result holds for $n = 2$.

Assume the result true for all $1 \leq n \leq k$, where $k \geq 2$. Now for $n = k+1$ we have $a_{k+1} = a_k + a_{k-1} < (7/4)^k + (7/4)^{k-1} = (7/4)^{k-1}[(7/4)+1] = (7/4)^{k-1}(11/4) = (7/4)^{k-1}(44/16) < (7/4)^{k-1}(49/16) = (7/4)^{k-1}(7/4)^2 = (7/4)^{k+1}$. So by the Alternative Form of the Principle of Mathematical Induction it follows that $a_n < (7/4)^n$ for all $n \geq 1$.

25.

$$\begin{aligned}
E(X) &= \textstyle\sum_x xPr(X=x) = \sum_{x=1}^n x(\frac{1}{n}) = (\frac{1}{n})\sum_{x=1}^n x = (\frac{1}{n})[\frac{n(n+1)}{2}] = \frac{n+1}{2}\\
E(X^2) &= \textstyle\sum_x x^2 Pr(X=x) = \sum_{x=1}^n x^2(\frac{1}{n}) = (\frac{1}{n})\sum_{x=1}^n x^2 = (\frac{1}{n})[\frac{n(n+1)(2n+1)}{6}] = \frac{(n+1)(2n+1)}{6}\\
\text{Var}(X) &= E(X^2) - E(X)^2 = \frac{(n+1)(2n+1)}{6} - \frac{(n+1)^2}{4} = (n+1)[\frac{2n+1}{6} - \frac{n+1}{4}]\\
&= (n+1)[\frac{4n+2-(3n+3)}{12}] = \frac{(n+1)(n-1)}{12} = \frac{n^2-1}{12}.
\end{aligned}$$

26.

a)
$$\begin{aligned}
a_1 &= \textstyle\sum_{i=0}^{1-1}\binom{0}{i}a_i a_{(1-1)-i} = \binom{0}{0}a_0 a_0 = a_0^2\\
a_2 &= \textstyle\sum_{i=0}^{2-1}\binom{1}{i}a_i a_{(2-1)-i} = \binom{1}{0}a_0 a_1 + \binom{1}{1}a_1 a_0 = 2a_0^3
\end{aligned}$$

b)
$$\begin{aligned}
a_3 &= \textstyle\sum_{i=0}^{3-1}\binom{3-1}{i}a_i a_{(3-1)-i} = \sum_{i=0}^{2}\binom{2}{i}a_i a_{2-i}\\
&= \binom{2}{0}a_0 a_2 + \binom{2}{1}a_1 a_1 + \binom{2}{2}a_2 a_0\\
&= a_0(2a_0^3) + 2(a_0^2)(a_0^2) + (2a_0^3)a_0 = 6a_0^4
\end{aligned}$$

c)
$$\begin{aligned}
a_4 &= \textstyle\sum_{i=0}^{4-1}\binom{4-1}{i}a_i a_{(4-1)-i} = \sum_{i=0}^{3}\binom{3}{i}a_i a_{3-i}\\
&= \binom{3}{0}a_0 a_3 + \binom{3}{1}a_1 a_2 + \binom{3}{2}a_2 a_1 + \binom{3}{3}a_3 a_0\\
&= a_0(6a_0^4) + 3(a_0^2)(2a_0^3) + 3(2a_0^3)(a_0^2) + (6a_0^4)(a_0)\\
&= 24a_0^5
\end{aligned}$$

c) For $n \geq 0$, $a_n = (n!)a_0^{n+1}$.

Proof: (By the Alternative Form of the Principle of Mathematical Induction)
The result is true for $n = 0$ and this establishes the basis step. [In fact, the calculations in parts (a) and (b) show the result is also true for $n = 1, 2, 3$, and 4.] Assuming the result true for $n = 0, 1, 2, 3, \ldots, k(\geq 0)$ – that is, that $a_n = (n!)a_0^{n+1}$ for $n = 0, 1, 2, 3, \ldots, k(\geq 0)$ – we find that

$$\begin{aligned} a_{k+1} &= \sum_{i=0}^{k} \binom{k}{i} a_i a_{k-i} \\ &= \sum_{i=0}^{k} \binom{k}{i} (i!)(a_0^{i+1})(k-i)!(a_0^{k-i+1}) \\ &= \sum_{i=0}^{k} \binom{k}{i} (i!)(k-i)! a_0^{k+2} \\ &= \sum_{i=0}^{k} k! a_0^{k+2} \\ &= (k+1)[k! a_0^{k+2}] = (k+1)! a_0^{k+2}. \end{aligned}$$

So the truth of the result for $n = 0, 1, 2, \ldots, k(\geq 0)$ implies the truth of the result for $n = k+1$. Consequently, for all $n \geq 0$, $a_n = (n!)a_0^{n+1}$ by the Alternative Form of the Principle of Mathematical Induction.

27. Let $T = \{n \in \mathbf{Z}^+ | n \geq n_0 \text{ and } S(n) \text{ is false}\}$. Since $S(n_0)$, $S(n_0+1)$, $S(n_0+2), \ldots, S(n_1)$ are true, we know that $n_0, n_0+1, n_0+2, \ldots, n_1 \notin T$. If $T \neq \emptyset$, then by the Well-Ordering Principle T has a least element r, because $T \subseteq \mathbf{Z}^+$. However, since $S(n_0)$, $S(n_0+1), \ldots, S(r-1)$ are true, it follows that $S(r)$ is true. Hence $T = \emptyset$ and the result follows.

28. (a) (i) The number of compositions of 5 that start with 1 is the number of compositions of 4, which is $2^{4-1} = 2^3 = 8$.

(ii) $2^{3-1} = 2^2 = 4$ (iii) $2^{2-1} = 2^1 = 2$
(iv) $2^{1-1} = 2^0 = 1$ (v) 1

(b) In total, there are $2^{(n+1)-1} = 2^n$ compositions for the fixed positive integer $n+1$. For $1 \leq i \leq n$, there are $2^{(n+1-i)-1} = 2^{n-i}$ compositions of $n+1$ that start with i. In addition, there is the composition consisting of only one summand – namely, $(n+1)$. So we have counted the same collection of objects – that is, the compositions of $n+1$ – in two ways. This gives us $2^n = \sum_{i=1}^{n} 2^{n-i} + 1 = (2^{n-1} + 2^{n-2} + 2^{n-3} + \cdots + 2^2 + 2^1 + 2^0) + 1 = (2^0 + 2^1 + 2^2 + \cdots + 2^{n-3} + 2^{n-2} + 2^{n-1}) + 1 = \sum_{i=0}^{n-1} 2^i + 1$. Consequently, $\sum_{i=0}^{n-1} 2^i = 2^n - 1$.

Section 4.2

1.

(a) $c_1 = 7$; and
$c_{n+1} = c_n + 7$, for $n \geq 1$.

(b) $c_1 = 7$; and
$c_{n+1} = 7c_n$, for $n \geq 1$.

(c) $c_1 = 10$; and
$c_{n+1} = c_n + 3$, for $n \geq 1$.

(d) $c_1 = 7$; and
$c_{n+1} = c_n$, for $n \geq 1$.

(e) $c_1 = 1$; and
$c_{n+1} = c_n + 2n + 1$, for $n \geq 1$.

(f) $c_1 = 3, c_2 = 1$; and
$c_{n+2} = c_n$, for $n \geq 1$.

2. (a) For any statements $p_1, p_2, \ldots, p_n, p_{n+1}$, we define

(1) the disjunction of p_1, p_2 as $p_1 \vee p_2$; and
(2) the disjunction of $p_1, p_2, \ldots, p_n, p_{n+1}$ by $p_1 \vee p_2 \vee \ldots \vee p_n \vee p_{n+1} \iff (p_1 \vee p_2 \vee \ldots \vee p_n) \vee p_{n+1}$.

(b) The result is true for $n = 3$. This is the Associative Law of $\vee$ of Section 2.2. Now assume the truth of the result for $n = k \geq 3$ and all $1 \leq r < k$, that is, $(p_1 \vee p_2 \vee \ldots \vee p_r) \vee (p_{r+1} \vee \ldots \vee p_k) \iff (p_1 \vee p_2 \vee \ldots \vee p_r \vee p_{r+1} \vee \ldots \vee p_k)$.
When we consider the case for $n = k+1$ we must account for all $1 \leq r < k+1$.

1) If $r = k$, then $(p_1 \vee p_2 \vee \ldots \vee p_k) \vee p_{k+1} \iff p_1 \vee p_2 \vee \ldots \vee p_k \vee p_{k+1}$, from our recursive definition.

2) For $1 \leq r < k$, we have $(p_1 \vee p_2 \vee \ldots \vee p_r) \vee (p_{r+1} \vee \ldots \vee p_k \vee p_{k+1}) \iff (p_1 \vee p_2 \vee \ldots \vee p_r) \vee [(p_{r+1} \vee \ldots \vee p_k) \vee p_{k+1}] \iff [(p_1 \vee p_2 \vee \ldots \vee p_r) \vee (p_{r+1} \vee \ldots \vee p_k)] \vee p_{k+1} \iff (p_1 \vee p_2 \vee \ldots \vee p_r \vee p_{r+1} \vee \ldots \vee p_k) \vee p_{k+1} \iff p_1 \vee p_2 \vee \ldots \vee p_r \vee p_{r+1} \vee \ldots \vee p_k \vee p_{k+1}$.
So the result is true for all $n \geq 3$ by the Principle of Mathematical Induction.

3. For $n \in \mathbf{Z}^+, n \geq 2$, let $T(n)$ denote the (open) statement: For the statements $p, q_1, q_2, \ldots, q_n$, $p \vee (q_1 \wedge \ldots \wedge q_n) \iff (p \vee q_1) \wedge (p \vee q_2) \wedge \ldots \wedge (p \vee q_n)$.
The statement $T(2)$ is true by virtue of the Distributive Law of $\vee$ over $\wedge$. Assuming $T(k)$, for $k \geq 2$, we now examine the situation for the statements $p, q_1, q_2, \ldots, q_k, q_{k+1}$. We find that $p \vee (q_1 \wedge q_2 \wedge \ldots \wedge q_k \wedge q_{k+1}) \iff p \vee [(q_1 \wedge q_2 \wedge \ldots \wedge q_k) \wedge q_{k+1}] \iff [p \vee (q_1 \wedge q_2 \wedge \ldots \wedge q_k)] \wedge (p \vee q_{k+1}) \iff [(p \vee q_1) \wedge (p \vee q_2) \wedge \ldots \wedge (p \vee q_k)] \wedge (p \vee q_{k+1}) \iff (p \vee q_1) \wedge (p \vee q_2) \wedge \ldots \wedge (p \vee q_k) \wedge (p \vee q_{k+1})$. It then follows by the Principle of Mathematical Induction that the statement $T(n)$ is true for all $n \geq 2$.

4. (a) For $n = 2$, the result is simply the DeMorgan Law $\neg(p_1 \vee p_2) \iff \neg p_1 \wedge \neg p_2$. Assuming the truth of the result for $n = k$, we find for $n = k+1$ that $\neg(p_1 \vee p_2 \vee \ldots p_k \vee p_{k+1}) \iff \neg[(p_1 \vee p_2 \vee \ldots \vee p_k) \vee p_{k+1}] \iff \neg(p_1 \vee p_2 \vee \ldots \vee p_k) \wedge \neg p_{k+1} \iff (\neg p_1 \wedge \neg p_2 \wedge \ldots \wedge \neg p_k) \wedge \neg p_{k+1} \iff \neg p_1 \wedge \neg p_2 \wedge \ldots \wedge \neg p_k \wedge \neg p_{k+1}$, so the result is true for all $n \geq 2$, by the Principle of Mathematical Induction.

(b) This result can be obtained from part (a) by a similar argument, or by the Principle of Duality for statements.

5. (a) (i) The intersection of A_1, A_2 is $A_1 \cap A_2$.

(ii) The intersection of $A_1, A_2, \ldots, A_n, A_{n+1}$ is given by $A_1 \cap A_2 \cap \ldots \cap A_n \cap A_{n+1} = (A_1 \cap A_2 \cap \ldots \cap A_n) \cap A_{n+1}$, the intersection of the *two* sets: $A_1 \cap A_2 \cap \ldots \cap A_n$ and A_{n+1}.

(b) Let $S(n)$ denote the given (open) statement. Then the truth of $S(3)$ follows from the Associative Law of $\cap$. Assuming $S(k)$ true for some $k \geq 3$ and all $1 \leq r < k$, consider the case for $k+1$ sets. Then we find that

1) For $r = k$ we have $(A_1 \cap A_2 \cap \ldots \cap A_k) \cap A_{k+1} = A_1 \cap A_2 \cap \ldots \cap A_k \cap A_{k+1}$. This follows from the given recursive definition.

2) For $1 \leq r < k$ we have $(A_1 \cap A_2 \cap \ldots \cap A_r) \cap (A_{r+1} \cap \ldots \cap A_k \cap A_{k+1}) = (A_1 \cap A_2 \cap \ldots \cap A_r) \cap [(A_{r+1} \cap \ldots \cap A_k) \cap A_{k+1}] = [(A_1 \cap A_2 \cap \ldots \cap A_r) \cap (A_{r+1} \cap \ldots \cap A_k)] \cap A_{k+1} = (A_1 \cap A_2 \cap \ldots \cap A_r \cap A_{r+1} \cap \ldots \cap A_k) \cap A_{k+1} = A_1 \cap A_2 \cap \ldots \cap A_r \cap A_{r+1} \cap \ldots \cap A_k \cap A_{k+1}$.
So by the Principle of Mathematical Induction, $S(n)$ is true for all $n \geq 3$.

6. (i) For $n = 2$, the result follows from the DeMorgan Laws. Assuming the result for $n = k \geq 2$, consider the case for $k + 1$ sets $A_1, A_2, \ldots, A_k, A_{k+1}$. Then $\overline{A_1 \cap A_2 \cap \ldots \cap A_k \cap A_{k+1}} = \overline{(A_1 \cap A_2 \cap \ldots \cap A_k) \cap A_{k+1}} = \overline{(A_1 \cap A_2 \cap \ldots \cap A_k)} \cup \overline{A_{k+1}} = [\overline{A_1} \cup \overline{A_2} \cup \ldots \cup \overline{A_k}] \cup \overline{A_{k+1}} = \overline{A_1} \cup \overline{A_2} \cup \ldots \cup \overline{A_k} \cup \overline{A_{k+1}}$, and the result is true for all $n \geq 2$, by the Principle of Mathematical Induction.

(ii) The proof for this result is similar to the one in part (i). – Simply replace each occurrence of $\cap$ by $\cup$, and vice versa. (We can also obtain (ii) from (i) by invoking the Principle of Duality – Theorem 3.5.)

7. For $n = 2$, the truth of the result $A \cap (B_1 \cup B_2) = (A \cap B_1) \cup (A \cap B_2)$ follows by virtue of the Distributive Law of $\cap$ over $\cup$.

Assuming the result for $n = k$, let us examine the case for the sets $A, B_1, B_2, \ldots, B_k, B_{k+1}$. We have $A \cap (B_1 \cup B_2 \cup \ldots \cup B_k \cup B_{k+1}) = A \cap [(B_1 \cup B_2 \cup \ldots \cup B_k) \cup B_{k+1}] = [A \cap (B_1 \cup B_2 \cup \ldots \cup B_k)] \cup (A \cap B_{k+1}) = [(A \cap B_1) \cup (A \cap B_2) \cup \ldots \cup (A \cap B_k)] \cup (A \cap B_{k+1}) = (A \cap B_1) \cup (A \cap B_2) \cup \ldots \cup (A \cap B_k) \cup (A \cap B_{k+1})$.

8. (a) (i) For $n = 2$, $x_1 + x_2$ denotes the ordinary sum of the real numbers x_1 and x_2.
(ii) For real numbers $x_1, x_2, \ldots, x_n, x_{n+1}$, we have $x_1 + x_2 + \ldots + x_n + x_{n+1} = (x_1 + x_2 + \ldots + x_n) + x_{n+1}$, the sum of the *two* real numbers $x_1 + x_2 + \ldots + x_n$ and x_{n+1}.

(b) The truth of this result for $n = 3$ follows from the Associative Law of Addition – since $x_1 + (x_2 + x_3) = (x_1 + x_2) + x_3$, there is no ambiguity in writing $x_1 + x_2 + x_3$.

Assuming the result true for all $k \geq 3$ and all $1 \leq r < k$, let us examine the case for $k + 1$ real numbers. We find that

1) When $r = k$ we have $(x_1 + x_2 + \ldots + x_r) + x_{r+1} = x_1 + x_2 + \ldots + x_r + x_{r+1}$, by virtue of the recursive definition.

2) For $1 \leq r < k$ we have $(x_1 + x_2 + \ldots + x_r) + (x_{r+1} + \ldots + x_k + x_{k+1}) = (x_1 + x_2 + \ldots + x_r) + [(x_{r+1} + \ldots + x_k) + x_{k+1}] = [(x_1 + x_2 + \ldots + x_r) + (x_{r+1} + \ldots + x_k)] + x_{k+1} = (x_1 + x_2 + \ldots + x_r + x_{r+1} + \ldots + x_k) + x_{k+1} = x_1 + x_2 + \ldots + x_r + x_{r+1} + \ldots + x_k + x_{k+1}$.

So the result is true for all $n \geq 3$ and all $1 \leq r < n$, by the Principle of Mathematical Induction.

9. a) (i) For $n = 2$, the expression x_1x_2 denotes the ordinary product of the real numbers x_1 and x_2.
(ii) Let $n \in \mathbf{Z}^+$ with $n \geq 2$. For the real numbers $x_1, x_2, \ldots, x_n, x_{n+1}$, we define

$$x_1x_2 \cdots x_nx_{n+1} = (x_1x_2 \cdots x_n)x_{n+1},$$

the product of the *two* real numbers $x_1x_2 \cdots x_n$ and x_{n+1}.

b) The result holds for $n = 3$ by the Associative Law of Multiplication (for real numbers). So $x_1(x_2x_3) = (x_1x_2)x_3$, and there is no ambiguity in writing $x_1x_2x_3$.
Assuming the result true for some (particular) $k \geq 3$ and all $1 \leq r < k$, let us examine the case for $k + 1$ (≥ 4) real numbers. We find that

1) When $r = k$ we have

$$(x_1x_2 \cdots x_r)x_{r+1} = x_1x_2 \cdots x_rx_{r+1}$$

by virtue of the recursive definition.

2) For $1 \le r < k$ we have

$$(x_1x_2\cdots x_r)(x_{r+1}\cdots x_kx_{k+1}) = (x_1x_2\cdots x_r)((x_{r+1}\cdots x_k)x_{k+1})$$

$= ((x_1x_2\cdots x_r)(x_{r+1}\cdots x_k))x_{k+1} = (x_1x_2\cdots x_rx_{r+1}\cdots x_k)x_{k+1}$
$= x_1x_2\cdots x_rx_{r+1}\cdots x_kx_{k+1}$,
so the result is true for all $n \ge 3$ and all $1 \le r < n$, by the Principle of Mathematical Induction.

10. The result is true for $n = 2$ by the material presented at the start of the problem. Assuming the truth for $n = k$ real numbers, we have, for $n = k+1, |x_1 + x_2 + \ldots + x_k + x_{k+1}| = |(x_1 + x_2 + \ldots + x_k) + x_{k+1}| \le |x_1 + x_2 + \ldots + x_k| + |x_{k+1}| \le |x_1| + |x_2| + \ldots + |x_k| + |x_{k+1}|$, so the result is true for all $n \ge 2$ by the Principle of Mathematical Induction.

11. Proof: (By the Alternative Form of the Principle of Mathematical Induction)
For $n = 0, 1, 2$ we have
$(n = 0)$ $a_{0+2} = a_2 = 1 \ge (\sqrt{2})^0$;
$(n = 1)$ $a_{1+2} = a_3 = a_2 + a_0 = 2 \ge \sqrt{2} = (\sqrt{2})^1$; and
$(n = 2)$ $a_{2+2} = a_4 = a_3 + a_1 = 2 + 1 = 3 \ge 2 = (\sqrt{2})^2$.
Therefore the result is true for these first three cases, and this gives us the basis step for the proof.
Next, for some $k \ge 2$ we assume the result true for all $n = 0, 1, 2, \ldots, k$. When $n = k + 1$ we find that
$a_{(k+1)+2} = a_{k+3} = a_{k+2} + a_k \ge (\sqrt{2})^k + (\sqrt{2})^{k-2} = [(\sqrt{2})^2 + 1](\sqrt{2})^{k-2} = 3(\sqrt{2})^{k-2}$
$= (3/2)(2)(\sqrt{2})^{k-2} = (3/2)(\sqrt{2})^k \ge (\sqrt{2})^{k+1}$, because $(3/2) = 1.5 > \sqrt{2}$ ($\doteq 1.414$). This provides the inductive step for the proof.
From the basis and inductive steps it now follows by the Alternative Form of the Principle of Mathematical Induction that $a_{n+2} \ge (\sqrt{2})^n$ for all $n \in \mathbf{N}$.

12. Proof: (By Mathematical Induction)
We find that $F_0 = \sum_{i=0}^{0} F_i = 0 = 1 - 1 = F_2 - 1$, so the given statement holds in this first case — and this provides the basis step of the proof.
For the inductive step we assume the truth of the statement when $n = k$ (≥ 0) — that is, that $\sum_{i=0}^{k} F_i = F_{k+2} - 1$. Now we consider what happens when $n = k + 1$. We find for this case that

$$\sum_{i=0}^{k+1} F_i = (\sum_{i=0}^{k} F_i) + F_{k+1} = (F_{k+2} - 1) + F_{k+1} = (F_{k+2} + F_{k+1}) - 1 = F_{k+3} - 1,$$

so the truth of the statement at $n = k$ implies the truth at $n = k + 1$.
Consequently, $\sum_{i=0}^{n} F_i = F_{n+2} - 1$ for all $n \in \mathbf{N}$ — by the Principle of Mathematical Induction.

13. Proof: (By Mathematical Induction).
Basis Step: When $n = 1$ we find that

$$\sum_{i=1}^{1} \frac{F_{i-1}}{2^i} = F_0/2 = 0 = 1 - (2/2) = 1 - \frac{F_3}{2} = 1 - \frac{F_{1+2}}{2^1},$$

so the result holds in the first case.
Inductive Step: Assuming the given (open) statement for $n = k$, we have $\sum_{i=1}^{k} \frac{F_{i-1}}{2^i} = 1 - \frac{F_{k+2}}{2^k}$.
When $n = k + 1$, we find that

$$\sum_{i=1}^{k+1} \frac{F_{i-1}}{2^i} = \sum_{i=1}^{k} \frac{F_{i-1}}{2^i} + \frac{F_k}{2^{k+1}} = 1 - \frac{F_{k+2}}{2^k} + \frac{F_k}{2^{k+1}}$$

$$= 1 + (1/2^{k+1})[F_k - 2F_{k+2}] = 1 + (1/2^{k+1})[(F_k - F_{k+2}) - F_{k+2}]$$

$$= 1 + (1/2^{k+1})[-F_{k+1} - F_{k+2}] = 1 - (1/2^{k+1})(F_{k+1} + F_{k+2}) = 1 - (F_{k+3}/2^{k+1}).$$

From the basis and inductive steps it follows from the Principle of Mathematical Induction that

$$\forall n \in \mathbf{Z}^+ \ \sum_{i=1}^{n}(F_{i-1}/2^i) = 1 - (F_{n+2}/2^n).$$

14. Proof: (By Mathematical Induction)
For $n = 1$ we find

$$L_1^2 = 1^2 = 1 = (1)(3) - 2 = L_1L_2 - 2,$$

so the result holds in this first case.
Next we assume the result is true when $n = k$. This gives us $\sum_{i=1}^{k} L_i^2 = L_kL_{k+1} - 2$. Then for $n = k+1$ we find that $\sum_{i=1}^{k+1} L_i^2 = \sum_{i=1}^{k} L_i^2 + L_{k+1}^2 = L_kL_{k+1} - 2 + L_{k+1}^2 = L_kL_{k+1} + L_{k+1}^2 - 2$
$= L_{k+1}(L_k + L_{k+1}) - 2 = L_{k+1}L_{k+2} - 2.$
Consequently, by the Principle of Mathematical Induction, it follows that

$$\forall n \in \mathbf{Z}^+ \ \sum_{i=1}^{n} L_i^2 = L_nL_{n+1} - 2.$$

15. Proof: (By the Alternative Form of the Principle of Mathematical Induction)
The result holds for $n = 0$ and $n = 1$ because
$(n = 0)$ $5F_{0+2} = 5F_2 = 5(1) = 5 = 7 - 2 = L_4 - L_0 = L_{0+4} - L_0$; and
$(n = 1)$ $5F_{1+2} = 5F_3 = 5(2) = 10 = 11 - 1 = L_5 - L_1 = L_{1+4} - L_1$.
This establishes the basis step for the proof.
Next we assume the induction hypothesis — that is, that for some k (≥ 1), $5F_{n+2} = L_{n+4} - L_n$ for all $n = 0, 1, 2, \ldots, k-1, k$. It then follows that for $n = k+1$,

$5F_{(k+1)+2} = 5F_{k+3} = 5(F_{k+2} + F_{k+1}) = 5(F_{k+2} + F_{(k-1)+2})$
$= 5F_{k+2} + 5F_{(k-1)+2} = (L_{k+4} - L_k) + (L_{(k-1)+4} - L_{k-1}) = (L_{k+4} - L_k) + (L_{k+3} - L_{k-1})$
$= (L_{k+4} + L_{k+3}) - (L_k + L_{k-1}) = L_{k+5} - L_{k+1} = L_{(k+1)+4} - L_{k+1}$ — where we have used the recursive definitions of the Fibonacci numbers and Lucas numbers to establish the second and eighth equalities.
It then follows by the Alternative Form of the Principle of Mathematical Induction that

$$\forall n \in \mathbf{N}\ \ 5F_{n+2} = L_{n+4} - L_n.$$

16. (a) Let E denote the set of all positive even integers. We define E recursively by
(1) $2 \in E$; and
(2) For each $n \in E$, $n + 2 \in E$.
(b) If G denotes the set of all nonnegative even integers we define G recursively by
(1) $0 \in G$; and
(2) For each $m \in G$, $m + 2 \in G$.

17.

(a)

Steps	**Reasons**
(1) p, q, r, T_0	Part (1) of the definition
(2) $(p \vee q)$	Step (1) and Part (2-ii) of the definition
(3) $(\neg r)$	Step (1) and Part (2-i) of the definition
(4) $(T_0 \wedge (\neg r))$	Steps (1), (3), and Part (2-iii) of the definition
(5) $((p \vee q) \rightarrow (T_0 \wedge (\neg r)))$	Steps (2), (4), and Part (2-iv) of the definition

(b)

Steps	**Reasons**
(1) p, q, r, s, F_0	Part (1) of the definition
(2) $(\neg p)$	Step (1) and Part (2-i) of the definition
(3) $((\neg p) \leftrightarrow q)$	Steps (1), (2), and Part (2-v) of the definition
(4) $(s \vee F_0)$	Step (1) and Part (2-ii) of the definition
(5) $(r \wedge (s \vee F_0))$	Steps (1), (4), and Part (2-iii) of the definition
(6) $(((\neg p) \leftrightarrow q) \rightarrow (r \wedge (s \vee F_0)))$	Steps (3), (5), and Part (2-iv) of the definition

18.

(a)	$k = 0:$	1	321
	$k = 1:$	4	132, 213, 231,312
	$k = 2:$	1	123
(b)	$k = 0:$	1	4321
	$k = 1:$	11	1432, 2143, 2431, 3142, 3214, 3241, 3421, 4132, 4213, 4231, 4312
	$k = 2:$	11	1243, 1324, 1342, 1423, 2134, 2314, 2341, 2413, 3124, 3412, 4123
	$k = 3:$	1	1 2 3 4

(c) Two descents

(d) $(m-1)-k = m-k-1$ descents.

(e) (i) Five locations: (1) In front of 1; (2) Between 1,2; (3) Between 2,4; (4) Between 3,6; (5) Between 5,8. [The *five* locations are determined by the *four* ascents and the *one* location at the start (in front of 1) of p.]

(ii) Four locations: (1) Between 4,3; (2) Between 6,5; (3) Between 8,7; (4) Following 7. [The *four* locations are determined by the *three* descents and the *one* location at the end (following 7) of p.]

(f) $\pi_{m,k} = (k+1)\pi_{m-1,k} + (m-k)\pi_{m-1,k-1}$.

Let $x: x_1, x_2, \ldots, x_m$ denote a permutation of $1, 2, 3, \ldots, m$ with k ascents (and $m-k-1$ descents). (1) If $m = x_m$ or if m occurs in $x_i m x_{i+2}$, $1 \leq i \leq m-2$, with $x_i > x_{i+2}$ then the removal of m results in a permutation of $1, 2, 3, \ldots, m-1$ with $k-1$ ascents – for a total of $[1+(m-k-1)]\pi_{m-1,k-1} = (m-k)\pi_{m-1,k-1}$ permutations. (2) If $m = x_1$ or if m occurs in $x_i m x_{i+2}$, $1 \leq i \leq m-2$, with $x_i < x_{i+2}$, then the removal of m results in a permutation of $1, 2, 3, \ldots, m-1$ with k ascents – for a total of $(k+1)\pi_{m-1,k}$ permutations.

Since cases (1) and (2) have nothing in common and account for all possibilities the recursive formula for $\pi_{m,k}$ follows. [Note: These are the Eulerian numbers $a_{m,k}$ of Example 4.21.]

19. (a) $\binom{k}{2} + \binom{k+1}{2} = [k(k-1)/2] + [(k+1)k/2] = (k^2 - k + k^2 + k)/2 = k^2$.

(c) $\binom{k}{3} + 4\binom{k+1}{3} + \binom{k+2}{3} = [k(k-1)(k-2)/6] + 4[(k+1)(k)(k-1)/6] + [(k+2)(k+1)(k)/6] = (k/6)[(k-1)(k-2) + 4(k+1)(k-1) + (k+2)(k+1)] = (k/6)[6k^2] = k^3$.

(d) $\sum_{k=1}^{n} k^3 = \sum_{k=1}^{n}\binom{k}{3} + 4\sum_{k=1}^{n}\binom{k+1}{3} + \sum_{k=1}^{n}\binom{k+2}{3} = \binom{n+1}{4} + 4\binom{n+2}{4} + \binom{n+3}{4} = (1/24)[(n+1)(n)(n-1)(n-2) + 4(n+2)(n+1)(n)(n-1) + (n+3)(n+2)(n+1)(n)] = [(n+1)(n)/24][(n-1)(n-2) + 4(n+2)(n-1) + (n+3)(n+2)] = [(n+1)(n)/24][6n^2+6n] = n^2(n+1)^2/4$.

(e) $k^4 = \binom{k}{4} + 11\binom{k+1}{4} + 11\binom{k+2}{4} + \binom{k+3}{4}$

In general, $k^t = \sum_{r=0}^{t-1} a_{t,r}\binom{k+r}{t}$, where the $a_{t,r}$'s are the Eulerian numbers of Example 4.21. [The given summation formula is known as Worpitzky's identity.]

20. (a) For $n = 2$, $[(p_1 \longrightarrow p_2) \wedge (p_2 \longrightarrow p_3)] \Longrightarrow [(p_1 \wedge p_2) \longrightarrow p_3]$, for if $(p_1 \wedge p_2) \longrightarrow p_3$ has value 0, then p_1 and p_2 have value 1 and p_3 has value 0. But then $p_2 \rightarrow p_3$ and $(p_1 \longrightarrow p_2) \wedge (p_2 \longrightarrow p_3)$ both have value 0. Assume the result for $n = k-1$, and consider the case of $n = k$. Then $[(p_1 \longrightarrow p_2) \wedge (p_2 \longrightarrow p_3) \wedge \ldots \wedge (p_{k-1} \longrightarrow p_k) \wedge (p_k \longrightarrow p_{k+1})] \Longrightarrow [((p_1 \wedge p_2 \wedge \ldots \wedge p_{k-1}) \longrightarrow p_k) \wedge (p_k \longrightarrow p_{k+1})] \Longrightarrow [(p_1 \wedge p_2 \wedge \ldots \wedge p_k) \longrightarrow p_{k+1}]$.

(b) Suppose that $S(1)$ is true and that if $S(k)$ is true for some $k \in \mathbf{Z}^+$, then $S(k) \Longrightarrow S(k+1)$. Then we find that $[S(1) \Longrightarrow S(2), S(2) \Longrightarrow S(3), \ldots, S(k) \Longrightarrow S(k+1)]$ and by part (a), $[(S(1) \wedge S(2) \wedge \ldots \wedge S(k)) \Longrightarrow S_{k+1}]$. So by Theorem 4.2, $S(n)$ is true for all n. Hence Theorem 4.2 implies Theorem 4.1.

(c) If $n = 1$ the result follows. Assume the result for $n = k$ (≥ 1), for some $k \in \mathbf{Z}^+$ and consider the case for $n = k+1$. If $1 \in S$ then the result follows. If $1 \notin S$, let

$T = \{x - 1 | x \in S\} \neq \emptyset$. Then $k \in T$ and by applying the induction hypothesis to T, T has a least element $t \geq 1$ and S has a least element $t + 1 \geq 2$.

(d) From part (c), Theorem 4.1 implies the Well-Ordering Principle. In the solution of Exercise 27 of Section 4.1 the Well-Ordering Principle implies Theorem 4.2. Hence Theorem 4.1 implies Theorem 4.2.

Section 4.3

1. (a) $a = a \cdot 1$, so $1|a; 0 = a \cdot 0$, so $a|0$.
(b) $a|b \Longrightarrow b = ac$, for some $c \in \mathbf{Z}$. $b|a \Longrightarrow a = bd$, for some $d \in \mathbf{Z}$. So $b = ac = b(dc)$ and $d = c = 1$ or -1. Hence $a = b$ or $a = -b$.
(c) $a|b \Longrightarrow b = ax, b|c \Longrightarrow c = by$, for some $x, y, \in \mathbf{Z}$. So $c = by = a(xy)$ and $a|c$.
(d) $a|b \Longrightarrow ac = b$, for some $c \in \mathbf{Z} \Longrightarrow acx = bx \Longrightarrow a|bx$.
(e) If $a|x, a|y$ then $x = ac, y = ad$ for some $c, d \in \mathbf{Z}$. So $z = x - y = a(c - d)$, and $a|z$. The proofs for the other cases are similar.
(g) Follows from part (f) by the Principle of Mathematical Induction.

2. (a) $a|b \Longrightarrow ax = b$, for some $x \in \mathbf{Z}^+$; $c|d \Longrightarrow cy = d$, for some $y \in \mathbf{Z}^+$. Then $(ac)(xy) = bd$, so $ac|bd$.
(c) $ac|bc \Longrightarrow acx = bc$, for some $x \in \mathbf{Z}^+ \Longrightarrow (ax - b)c = 0 \Longrightarrow [ax - b = 0$, since $c > 0] \Longrightarrow ax = b \Longrightarrow a|b$.
The proof for part (b) is similar.

3. Since q is prime its only positive divisors are 1 and q. With p a prime, $p > 1$. Hence $p|q \Longrightarrow p = q$.

4. No. $6|(2 \cdot 3)$ but $6 \not| 2$ and $6 \not| 3$.

5. Proof: (By the Contrapositive)
Suppose that $a \mid b$ or $a \mid c$.
If $a \mid b$, then $ak = b\ \exists k \in \mathbf{Z}$. But $ak = b \Rightarrow (ak)c = a(kc) = bc \Rightarrow a \mid bc$.
A similar result is obtained if $a \mid c$.

6. Proof: (By Mathematical Induction)
The result for $n = 2$ is true by virtue of part (a) of Exercise 2. So assume the result for $n = k\ (\geq 2)$. Then consider the case for $n = k + 1$: We have positive integers $a_1, a_2, \ldots, a_k, a_{k+1}$, $b_1, b_2, \ldots, b_k, b_{k+1}$, where $a_i|b_i$ for all $1 \leq i \leq k + 1$. If we let $a = a_1a_2 \cdots a_k$ and $b = b_1b_2 \cdots b_k$, then we know that $a|b$ by the induction hypothesis. And since $a|b$ and $a_{k+1}|b_{k+1}$, by the case for $n = 2$, it follows that $a \cdot a_{k+1}|b \cdot b_{k+1}$, or $(a_1 \cdot a_2 \cdots a_k \cdot a_{k+1})|(b_1 \cdot b_2 \cdots b_k \cdot b_{k+1})$.
Therefore the result is true for all $n \geq 2$ — by the Principle of Mathematical Induction.

7. a) Let $a = 1$, $b = 5$, $c = 2$. Another example is $a = b = 5$, $c = 3$.

b) Proof:
$31|(5a+7b+11c) \Longrightarrow 31|(10a+14b+22c)$. Also, $31|(31a+31b+31c)$, so $31|[(31a+31b+31c)-(10a+14b+22c)]$. Hence $31|(21a+17b+9c)$.

8. Note that each of Eleanor's 12 numbers is divisible by 6. Consequently, every sum that uses any of these numbers must also be divisible by 6 (because of part (g) of Theorem 4.3 – where each $x_i = 1$, for $1 \le i \le n$). Unfortunately 500 is not divisible by 6, so Eleanor has not received a winning card.

9. $b|a, b|(a+2) \Longrightarrow b|[ax+(a+2)y]$ for all $x, y \in \mathbf{Z}$. Let $x=-1, y=1$. Then $b>0$ and $b|2$, so $b=1$ or 2.

10. Let $n=2k+1, k \ge 0$. $n^2-1=(2k+1)^2-1=4k^2+4k=4k(k+1)$. Since one of $k, k+1$ must be even, it follows that $8|(n^2-1)$.

11. Let $a=2m+1, b=2n+1$, for some $m, n \ge 0$. Then $a^2+b^2=4(m^2+m+n^2+n)+2$, so $2|(a^2+b^2)$ but $4 \nmid (a^2+b^2)$.

12. (a) $23=3\cdot 7+2, \quad q=3, r=2$.
(b) $-115=(-10)\cdot 12+5, \quad q=-10, r=5$.
(c) $0=0\cdot 42+0, \quad q=0, r=0$.
(d) $434=14\cdot 31+0, \quad q=14, r=0$.

13. Proof:
For $n=0$ we have $7^n-4^n=7^0-4^0=1-1=0$, and $3|0$. So the result is true for this first case. Assuming the truth for $n=k$ we have $3|(7^k-4^k)$. Turning to the case for $n=k+1$ we find that $7^{k+1}-4^{k+1}=7(7^k)-4(4^k)=(3+4)(7^k)-4(4^k)=3(7^k)+4(7^k-4^k)$. Since $3|3$ and $3|(7^k-4^k)$ (by the induction hypothesis), it follows from part (f) of Theorem 4.3 that $3|[3(7^k)+4(7^k-4^k)]$, that is, $3|(7^{k+1}-4^{k+1})$. It now follows by the Principle of Mathematical Induction that $3|(7^n-4^n)$ for all $n \in \mathbf{N}$.

14. (a) $137=(10001001)_2=(2021)_4=(211)_8$
(b) $6243=(1100001100011)_2=(1201203)_4=(14143)_8$
(c) $12,345=(11000000111001)_2=(3000321)_4=(30071)_8$

15.

	Base 10	Base 2	Base 16
(a)	22	10110	16
(b)	527	1000001111	20F
(c)	1234	10011010010	4D2
(d)	6923	1101100001011	1B0B

16.

	Base 16	Base 2	Base 10
(a)	A7	10100111	167
(b)	4C2	10011000010	1218
(c)	1CB2	1110010110010	7346
(d)	A2DFE	10100010110111111110	667134

17.

	Base 2	Base 10	Base 16
(a)	11001110	206	CE
(b)	00110001	49	31
(c)	11110000	240	F0
(d)	01010111	87	57

18. The base 7.

19. Here n is a divisor of 18 – so $n \in \{1, 2, 3, 6, 9, 18\}$.

20. (a) 00001111 (b) 11110001 (c) 01100100

(d) Start with the binary representation of 65

$$65 \downarrow 01000001$$

Interchanges the 0's and 1's to obtain the one's complement

$$\downarrow 10111110$$

Add 1 to the one's complement

$$\downarrow 10111111$$

(e) 01111111 (f) 10000000

21.

	Largest Integer	Smallest Integer
(a)	$7 = 2^3 - 1$	$-8 = -(2^3)$
(b)	$127 = 2^7 - 1$	$-128 = -(2^7)$
(c)	$2^{15} - 1$	$-(2^{15})$
(d)	$2^{31} - 1$	$-(2^{31})$
(e)	$2^{n-1} - 1$	$-(2^{n-1})$

22.

(a)
$$\begin{array}{rl} 0101 & (= 5) \\ \underline{+0001} & (= 1) \\ 0110 & (= 6) \end{array}$$

(b)
$$\begin{array}{rl} 1101 & (= -3) \\ \underline{+1110} & (= -2) \\ 1011 & (= -5) \end{array}$$

(c)
$$\begin{array}{rl} 0111 & (= 7) \\ \underline{+1000} & (= -8) \\ 1111 & (= -1) \end{array}$$

(d)
$$\begin{array}{rl} 1101 & (= -3) \\ \underline{+1010} & (= -6) \\ 0111 & (\neq -9) \text{ (overflow error)} \end{array}$$

23. $ax = ay \implies ax - ay = 0 \implies a(x - y) = 0$. In the system of integers, if $b, c \in \mathbf{Z}$ and $bc = 0$, then $b = 0$ or $c = 0$. Since $a(x - y) = 0$ and $a \neq 0$ then $(x - y) = 0$ and $x = y$.

24.

```
Program   ChangeOfBase (Input,Output);
Var Number, Base, Remainder,
     Power, Result, Keep : Integer;
Begin
     Writeln ('Input the base 10 number - positive integer - that is to be changed.');
     Write ('Number = ');
     Read (Number);
     Writeln ('Input the base - an integer between 2 and 9 inclusive.');
     Write (' Base = ');
     Read (Base);
     Keep := Number;
     Result := 0;
     Power := 1;
     While Number > 0 Do
          Begin
               Remainder := Number Mod Base;
               Result := Result + (Remainder * Power);
               Power := Power * 10;
               Number := Number Div Base
          End;
     Writeln ('The number ', Keep:0, 'when converted to',
          'base ', Base:0, 'is written as ', Result:0)
End.
```

25. (i) If $a = 0$, choose $q = r = 0$.
(ii) Let $a > 0, b < 0$. Then $-b > 0$ so there exist $q, r \in \mathbf{Z}$ with $a = q(-b) + r$, where $0 \le r < (-b)$. Hence $a = (-q)b + r$ with $0 \le r < |b|$.
(iii) Finally, consider the case where $a < 0$ and $b < 0$. Then $-a, -b > 0$ so $-a = q'(-b) + r'$ with $0 \le r' < (-b)$. So $a = q'b - r' = (q' + 1)b + (-r' - b) = qb + r$ with $0 \le r = -b - r' < -b = |b|$.
For uniqueness, let $a, b, \in \mathbf{Z}, b \ne 0$, and assume $a = q_1b + r_1 = q_2b + r_2$, where $0 \le r_1, r_2 \le |b|$. Then $0 = (q_1 - q_2)b + (r_1 - r_2)$ and $|q_1 - q_2||b| = |r_1 - r_2|$. If $r_1 \ne r_2$, then $|r_1 - r_2| > 0$ but $|r_1 - r_2| < |b|$. Hence $|q_1 - q_2||b| < |b|$. This can only happen if $q_1 = q_2$. But then $r_1 = r_2$, so q, r are unique.

26.

```
Program Base_16(Input,Output);
(* This program converts a positive integer less than 4,294,967,295 (= 16^8 - 1) to base
16.*)

Type
   sub1 = 0..15;
```

```
    sub2 = 10..15;
    sub3 = 0..8;
    sub4 = -1..7;
Var
    remainders: sub1;
    larger: sub2;
    positions: array [0..7] of sub 1;
    i: sub3;
    j: sub4;
    m,n: integer;
Begin
    Writeln ('What positive integer do you wish to convert to base 16?');
    Readln (n);
    For i := 0 to 7 do
        positions [i] := 0;
    m := n;
    i := 0;
    While m > 0 do
        Begin
            positions[i] := m mod 16;
            m := m div 16;
            i := i+1
        End;
    j := i-1;
    Write ('The integer  ', n:0, '  in base 16 is written  ');
    While j  >= 0 do
        Begin
            If positions[j] < 10 then
                Write (positions[j] : 1)
            Else
                Begin
                    larger := positions[j];
                    Case larger of
                        10: Write ('A');
                        11: Write ('B');
                        12: Write ('C');
                        13: Write ('D');
                        14: Write ('E');
                        15: Write ('F')
                    End
                End;
            j := j - 1
        End;
```

```
    Writeln ('.')
End.
```

27.

```
Program Divisors (input,output);
Var
    N, Divisor: Integer;
Begin
    Write ('The positive integer N whose divisors are sought is N =  ');
    Read (N);
    Writeln;
    If N = 1 Then
        Writeln ('The only divisor of 1 is 1.')
    Else
        Begin
        Writeln ('The divisors of ', N:0, 'are :');
        Writeln (1:8);
        If N Mod 2 = 0 Then
            Begin
                For Divisor := 2 to N Div 2 Do
                If N Mod Divisor = 0 Then
                    Writeln (Divisor:8)
            End
        Else
            For Divisor := 3 to N Div 3 Do
                If N Mod Divisor = 0 Then
                    Writeln (Divisor:8)
        End;
    Writeln (N:8)
End.
```

28. Proof: Let $Y = \{3k \mid k \in \mathbf{Z}^+\}$, the set of all positive integers divisible by 3. In order to show that $X = Y$ we shall verify that $X \subseteq Y$ and $Y \subseteq X$.
(i) ($X \subseteq Y$): By part (1) of the recursive definition of X we have 3 in X. And since $3 = 3 \cdot 1$, it follows that 3 is in Y. Turning to part (2) of this recursive definition suppose that for $x, y \in X$ we also have $x, y \in Y$. Now $x + y \in X$ by the definition and we need to show that $x + y \in Y$. This follows because $x, y \in Y \Rightarrow x = 3m, y = 3n$ for some $m, n \in \mathbf{Z}^+ \Rightarrow x + y = 3m + 3n = 3(m+n)$, with $m + n \in \mathbf{Z}^+ \Rightarrow x + y \in Y$. Therefore every positive integer that results from either part (1) or part (2) of the recursive definition of X is an element in Y, and, consequently, $X \subseteq Y$.

(ii) ($Y \subseteq X$): In order to establish this inclusion we need to show that every positive integer multiple of 3 is in X. This will be accomplished by the Principle of Mathematical Induction.
Start with the open statement

$$S(n):\ 3n \text{ is an element in } X,$$

which is defined for the universe $\mathbf{Z}^+$. The basis step — that is, $S(1)$ — is true because $3 \cdot 1 = 3$ is in X by part (1) of the recursive definition of X. For the inductive step of this proof we assume the truth of $S(k)$ for some k (≥ 1) and consider what happens at $n = k+1$. From the inductive hypothesis $S(k)$ we know that $3k$ is in X. Then from part (2) of the recursive definition of X we find that $3(k+1) = 3k+3 \in X$ because $3k,\ 3 \in X$. Hence $S(k) \Rightarrow S(k+1)$. So by the Principle of Mathematical Induction it follows that $S(n)$ is true for all $n \in \mathbf{Z}^+$ — and, consequently, $Y \subseteq X$.
With $X \subseteq Y$ and $Y \subseteq X$ it follows that $X = Y$.

29. (a) Since $2|10^t$ for all $t \in \mathbf{Z}^+, 2|n$ iff $2|r_0$. (b) Follows from the fact that $4|10^t$ for $t \geq 2$. (c) Follows from the fact that $8|10^t$ for $t \geq 3$.
In general, $2^{t+1}|n$ iff $2^{t+1}|(r_t \cdot 10^t + \cdots + r_1 \cdot 10 + r_0)$.

Section 4.4

1. (a) $1820 = 7(231) + 203$
$231 = 1(203) + 28$
$203 = 7(28) + 7$
$28 = 7(4)$, so gcd(1820,23) = 7
$7 = 203 - 7(28) = 203 - 7[231 - 203] = (-7)(231) + 8(203) = (-7)(231) + 8[1820 - 7(231)] = 8(1820) + (-63)(231)$
(b) gcd(1369,2597) = 1 = 2597(534) + 1369(-1013)
(c) gcd(2689,4001) = 1 = 4001(-1117) + 2689(1662)

2. (a) If $as + bt = 2$, then $\gcd(a, b) = 1$ or 2, for the gcd of a, b divides a, b so it divides $as + bt = 2$.
(b) $as + bt = 3 \Longrightarrow \gcd(a, b) = 1$ or 3.
(c) $as + bt = 4 \Longrightarrow \gcd(a, b) = 1, 2$ or 4.
(d) $as + bt = 6 \Longrightarrow \gcd(a, b) = 1, 2, 3$ or 6.

3. $\gcd(a, b) = d \Longrightarrow d = ax + by$, for some $x, y \in \mathbf{Z}$. $\gcd(a, b) = d \Longrightarrow a/d, b/d \in \mathbf{Z}$. $1 = (a/d)x + (b/d)y \Longrightarrow \gcd(a/d, b/d) = 1$.

4. Let $\gcd(a, b) = g$, $\gcd(na, nb) = h$. $\gcd(a, b) = g \Longrightarrow g = as + bt$, for some $s, t \in \mathbf{Z}$. $ng = (na)s + (nb)t$, so $h|ng$. $h = \gcd(na, nb) \Longrightarrow h = (na)x + (nb)y$, for some $x, y \in \mathbf{Z}$. $h = n(ax + by) \Longrightarrow n|h \Longrightarrow nh_1 = h$ for some $h_1 \in \mathbf{Z}$ and $h_1 = ax + by$. $g = \gcd(a, b) \Longrightarrow g|h_1 \Longrightarrow n(gh_2) = h$ for some $h_2 \in \mathbf{Z}$. Since $(ng)|h$ and $h|(ng)$, with $h, ng \in \mathbf{Z}^+$, it follows that $\gcd(na, nb) = h = ng = n\gcd(a, b)$.

5. Proof: Since $c = \gcd(a, b)$ we have $a = cx$, $b = cy$ for some $x, y \in \mathbf{Z}^+$. So $ab = (cx)(cy) = c^2(xy)$, and c^2 divides ab.

6. (a) $2 = 1(n+2) + (-1)n$. Since $\gcd(n, n+2)$ is the smallest positive integer that can be expressed as a linear combination of n and $n+2$, it follows that $\gcd(n, n+2) \leq 2$. Furthermore, $\gcd(n, n+2)|2$. Hence $\gcd(n, n+2) = 1$ or 2. In fact, $\gcd(n, n+2) = 1$, for n odd, and $\gcd(n, n+2) = 2$, for n even.

(b) Arguing as in part (a) we have $\gcd(n, n+3) = 1$ or 3. When n is a multiple of 3, then $\gcd(n, n+3) = 3$; otherwise, $\gcd(n, n+3) = 1$.

For $\gcd(n, n+4)$ we need to be cautious. The answer is not 1 or 4. Here we have $\gcd(n, n+4) = 1$ or 2 or 4. For n a multiple of 4, $\gcd(n, n+4) = 4$. When $n = 4t+2$, $t \in \mathbf{Z}^+$, we find that $\gcd(n, n+4) = 2$. For n odd, $\gcd(n, n+4) = 1$.

(c) In general, for $n, k \in \mathbf{Z}^+$, $\gcd(n, n+k)$ is a divisor of k. Consequently, if k is a prime, then $\gcd(n, n+k) = k$, for n a multiple of k, and $\gcd(n, n+k) = 1$, for n not a multiple of k.

7. Let $\gcd(a, b) = h$, $\gcd(b, d) = g$. $\gcd(a, b) = h \Longrightarrow h|a, h|b \Longrightarrow h|(a \cdot 1 + bc) \Longrightarrow h|d$. $h|b, h|d \Longrightarrow h|g$. $\gcd(b, d) = g \Longrightarrow g|b, g|d \Longrightarrow g|(d \cdot 1 + b(-c)) \Longrightarrow g|a$. $g|b, g|a, h = \gcd(a, b) \Longrightarrow g|h$. $h|g, g|h$, with $g, h \in \mathbf{Z}^+ \Longrightarrow g = h$.

8. $\gcd(a, b) = 1 \Longrightarrow ax + by = 1$ for some $x, y \in \mathbf{Z}$. Then $c = acx + bcy$. $a|c \Longrightarrow c = ad, b|c \Longrightarrow c = be$, so $c = ab(ex + dy)$ and $ab|c$. The result is false if $\gcd(a, b) \neq 1$. For example, let $a = 12, b = 18, c = 36$. Then $a|c, b|c$ but $(ab) \not| c$.

9. (a) If $c \in \mathbf{Z}^+$, then $c = \gcd(a, b)$ if (and only if)
(1) $c \mid a$ and $c \mid b$; and
(2) $\forall d \in \mathbf{Z}\ [[(d \mid a) \wedge (d \mid b)] \Rightarrow d \mid c]$

(b) If $c \in \mathbf{Z}^+$, then $c \neq \gcd(a, b)$ if (and only if)
(1) $c \not| a$ or $c \not| b$; or
(2) $\exists d \in \mathbf{Z}\ [(d \mid a) \wedge (d \mid b) \wedge (d \not| c)]$.

10. If $c = \gcd(a-b, a+b)$ then $c|[(a-b)x + (a+b)y]$ for all $x, y \in \mathbf{Z}$. In particular, for $x = y = 1, c|2a$, and for $x = -1, y = 1, c|2b$. From Exercise 4, $\gcd(2a, 2b) = 2\gcd(a, b) = 2$, so $c|2$ and $c = 1$ or 2.

11. $\gcd(a, b) = 1 \Longrightarrow ax + by = 1$, for some $a, b \in \mathbf{Z}$. Then $acx + bcy = c$. $a|acx, a|bcy$ (since $a|bc) \Longrightarrow a|c$.

12. Proof: Let $d_1 = \gcd(a, b)$ and $d_2 = \gcd(a-b, b)$.
$d_2 = \gcd(a-b, b) \Rightarrow [d_2|(a-b) \wedge d_2|b] \Rightarrow [d_2|[(a-b)+b]]$ by part (f) of Theorem 4.3 $\Rightarrow d_2|a$, and $[d_2|a \wedge d_2|b] \Rightarrow d_2|d_1$.

$d_1 = \gcd(a, b) \Rightarrow [d_1|a \wedge d_1|b] \Rightarrow d_1|[a+(-1)b]$, by part (f) of Theorem 4.3. Hence $d_1|(a-b)$.

Since $d_1|(a-b)$ and $d_1|b$, it follows that $d_1|d_2$. Consequently, we find that $[d_1|d_2 \wedge d_2|d_1 \wedge \gcd(d_1, d_2) > 0] \Rightarrow \gcd(a, b) = d_1 = d_2 = \gcd(a-b, b)$.

13. Proof: We find that for each $n \in \mathbf{Z}^+$, $(5n+3)(7)+(7n+4)(-5) = (35n+21)-(35n+20) = 1$. Consequently, it follows that the $\gcd(5n+3, 7n+4) = 1$, or $5n+3$ and $7n+4$ are relatively prime.

14. $33x + 29y = 2490$
$\gcd(33, 29) = 1$, and $33 = (1)(29) + 4, 29 = (7)(4) + 1$, so $1 = 29 - 7(4) = 29 - 7[33 - 29] = 8(29) - 7(33)$. $1 = 33(-7) + 29(8) \Longrightarrow 2490 = 33(-17430) + 29(19920) = 33(-17430 + 29k) + 29(19920 - 33k)$, for all $k \in \mathbf{Z}$.

$x = -17430 + 29k, y = 19920 - 33k$

$x \geq 0 \Longrightarrow 29k \geq 17430 \Longrightarrow k \geq 602$

$y \geq 0 \Longrightarrow 19920 \geq 33k \Longrightarrow 603 \geq k$

$k = 602 : x = 28, y = 54; k = 603 : x = 57, y = 21$.

15. We need to find $x, y \in \mathbf{Z}^+$ where $y > x$ and $20x + 50y = 1020$, or $2x + 5y = 102$. As $\gcd(2, 5) = 1$ we start with $2(-2) + 5(1) = 1$ and find that $2(-2) + 5(1) = 1 \Rightarrow 102 = 2(-204) + 5(102) = 2[-204 + 5k] + 5[102 - 2k]$. Since $x = -204 + 5k > 0$, it follows that $k > 204/5 = 40.8$ and $y = 102 - 2k > 0$ implies that $51 > k$. Consequently $k = 41, 42, 43, \ldots, 50$. Since $y > x$ we find the following solutions:

k	$x = -204 + 5k$	$y = 102 - 2k$
41	1	20
42	6	18
43	11	16

16. Proof: Suppose that there exist $c, d \in \mathbf{Z}^+$ with $cd = a$ and $\gcd(c, d) = b$. Since $\gcd(c, d) = b$, we have $c = bc_1$, $d = bd_1$. Consequently, $a = cd = (bc_1)(bd_1) = b^2(c_1 d_1)$, so $b^2|a$.

Conversely, $b^2|a \Rightarrow b^2x = a$, for some $x \in \mathbf{Z}^+$. Let $c = bx$ and $d = b$. Then $cd = a$ and $\gcd(c, d) = \gcd(bx, b) = b$.

17. $\gcd(84, 990) = 6$, so $84x + 990y = c$ has a solution x_0, y_0 in $\mathbf{Z}$ if $6|c$. For $10 < c < 20, 6|c \Longrightarrow c = 12$ or 18. There is no solution for $c = 11, 13, 14, 15, 16, 17, 19$.

When $c = 12, 84x + 990y = 12$ (or, $14x + 165y = 2$).

$165 = 11(14) + 11$

$14 = 1(11) + 3$

$11 = 3(3) + 2$

$3 = 1(2) + 1$

Therefore $1 = 3 - 2 = 3 - [11 - 3(3)] = 4(3) - 11 = 4[14 - 11] - 11 = 4(14) - 5(11) = 4(14) - 5[165 - 11(14)] = 59(14) - 5(165)$

$1 = 14(59) + 165(-5)$

$2 = 14(118) + 165(-10) = 14(118 - 165k) + 165(-10 + 14k)$.

The solutions for $84x + 990y = 12$ are $x = 118 - 165k, y = -10 + 14k, k \in \mathbf{Z}$.
When $c = 18$, the solutions are $x = 177 - 165k, y = -15 + 14k, k \in \mathbf{Z}$.

18. Let $a, b, c \in \mathbf{Z}^+$. If $ax + by = c$ has a solution $x_0, y_0 \in \mathbf{Z}$, then $ax_0 + by_0 = c$, and since $\gcd(a, b)$ divides a and b, $\gcd(a, b)|c$. Conversely, suppose $\gcd(a, b)|c$. Then $c = \gcd(a, b)d$, for some $d \in \mathbf{Z}$. Since $\gcd(a, b) = as + bt$, for some $s, t \in \mathbf{Z}$, we have $a(sd) + b(td) = \gcd(a, b)d = c$ or $ax_0 + by_0 = c$, and $ax + by = c$ has a solution in $\mathbf{Z}$.

Let $\gcd(a, b) = g$, $\text{lcm}(a, b) = h$. $\gcd(a, b) = g \Longrightarrow as + bt = g$, for some $s, t \in \mathbf{Z}$. $\text{lcm}(a, b) = h \Longrightarrow h = ma = nb$, for some $m, n \in \mathbf{Z}^+$. $hg = has + hbt = nbas + mabt = ab(ns + mt) \Longrightarrow ab|hg$. $\gcd(a, b) = g \Longrightarrow g|a, g|b$, so $(a/g)b = (b/g)a$ is a common multiple of a and b. Consequently $h|(a/g)b$, and $hx = (a/g)b$, for some $x \in \mathbf{Z}$, or $ghx = ab$. Hence $gh|ab$.

19. From Theorem 4.10 we know that $ab = \text{lcm}(a, b) \cdot \gcd(a, b)$. Consequently,
$b = [\text{lcm}(a, b) \cdot \gcd(a, b)]/a = (242,500)(105)/630 = 40,425$.

20. $\text{lcm}(a, b) = (ab)/\gcd(a, b)$
(a) lcm(231,1820) = (231) (1820)/7 = 60,060
(b) lcm(1369,2597) = (1369)(2597) = 3,555,293
(c) lcm(2689,4001) = (2689)(4001) = 10,758,689

21. $\gcd(n, n + 1) = 1$, $\text{lcm}(n, n + 1) = n(n + 1)$

22. Proof: The result follows from Theorem 14.10 and Exercise 4 for this section. We find that $\text{lcm}\ (na, nb) = \frac{(na)(nb)}{\gcd(na,nb)} = \frac{n^2 ab}{n \gcd(a,b)} = n[\frac{ab}{\gcd(a,b)}] = n\text{lcm}(a, b)$.

Section 4.5

1. (a) $2^2 \cdot 3^3 \cdot 5^3 \cdot 11$ (b) $2^4 \cdot 3 \cdot 5^2 \cdot 7^2 \cdot 11^2$
(c) $3^2 \cdot 5^3 \cdot 7^2 \cdot 11 \cdot 13$

2. $\gcd(148500, 7114800) = 2^2 \cdot 3 \cdot 5^2 \cdot 11 = 3300$
$\text{lcm}(148500, 7114800) = 2^4 \cdot 3^3 \cdot 5^3 \cdot 7^2 \cdot 11^2 = 320166000$
$\gcd(148500, 7882875) = 3^2 \cdot 5^3 \cdot 11 = 12375$
$\text{lcm}(148500, 7882875) = 2^2 \cdot 3^3 \cdot 5^3 \cdot 7^2 \cdot 11 \cdot 13 = 94594500$
$\gcd(7114800, 7882875) = 3 \cdot 5^2 \cdot 7^2 \cdot 11 = 40425$
$\text{lcm}(7114800, 7882875) = 2^4 \cdot 3^2 \cdot 5^3 \cdot 7^2 \cdot 11^2 \cdot 13 = 1387386000$

3.

$$m^2 = p_1^{2e_1} p_2^{2e_2} p_3^{2e_3} \cdots p_t^{2e_t}$$
$$m^3 = p_1^{3e_1} p_2^{3e_2} p_3^{3e_3} \cdots p_t^{3e_t}$$

4. The result is true for $n = 1$. From Lemma 4.2 the result follows for $n = 2$. For $k \geq 2$, assume that $p|a_1 a_2 \cdots a_k \Longrightarrow p|a_i$, for some $1 \leq i \leq k$. Now consider $p|a_1 a_2 \cdots a_k a_{k+1}$. Then $p|(a_1 a_1 \cdots a_k)a_{k+1} \Longrightarrow p|a_1 a_2 \cdots a_k$ or $p|a_{k+1}$ (by the case where $n = 2$) $\Longrightarrow p|a_i$ for some $1 \leq i \leq k$ (by the induction hypothesis) or $p|a_{k+1} \Longrightarrow p|a_i$ for some $1 \leq i \leq k+1$. The general result then follows by the Principle of Mathematical Induction.

5. Proof: (The proof is similar to that given in Example 4.41.)
If not, we have $\sqrt{p} = a/b$, where $a, b \in \mathbf{Z}^+$ and $\gcd(a,b) = 1$. Then $\sqrt{p} = a/b \Rightarrow p = a^2/b^2 \Rightarrow pb^2 = a^2 \Rightarrow p \mid a^2 \Rightarrow p \mid a$ (by Lemma 4.2). Since $p \mid a$ we know that $a = pk$ $\exists k \in \mathbf{Z}^+$, and $pb^2 = a^2 = (pk)^2 = p^2k^2$, or $b^2 = pk^2$. Hence $p \mid b^2$ and so $p \mid b$. But if $p \mid a$ and $p \mid b$ then $\gcd(a,b) = p > 1$ — contradicting our earlier claim that $\gcd(a,b) = 1$.

6. Here $25n + 10n + 40n = 100k$, so $75n = 100k$, or $3n = 4k$. From Lemma 4.2 it follows that $3|k$. So $k = 3 \cdot r$. Then $3n = 4(3 \cdot r) \Rightarrow n = 4r$. So n is any positive multiple of 4.

7. (a) $3 \times 4 \times 4 \times 2 = 96$ (b) 270 (c) 144

8. a) There are $(15)(10)(9)(11)(4)(6)(11) = 3{,}920{,}400$ positive divisors of $n = 2^{14}3^9 5^8 7^{10} 11^3 13^5 37^{10}$.
b) (i) $(14-3+1)(9-4+1)(8-7+1)(10-0+1)(3-2+1)(5-0+1)(10-2+1) = (12)(6)(2)(11)(2)(6)(9) = 171,072$
(ii) Since $1,166,400,000 = 2^9 3^6 5^5$, the number of divisors here is $(14-9+1)(9-6+1)(8-5+1)(10-0+1)(3-0+1)(5-0+1)(10-0+1) = (6)(4)(4)(11)(4)(6)(11) = 278,784$.
(iii) $(8)(5)(5)(6)(2)(3)(6) = 43{,}200$
(iv) $(7)(3)(4)(6)(1)(3)(6) = 9072$
(v) $(5)(4)(3)(4)(2)(2)(4) = 3840$
(vi) $(1)(1)(2)(2)(1)(1)(3) = 12$
(vii) $(3)(2)(2)(2)(1)(1)(2) = 48$

9. From Theorem 4.10 we know that $mn = \text{lcm}(m,n) \cdot \gcd(m,n)$, so $\gcd(m,n) = mn/\text{lcm}(m,n) = 2^2 3^1 5^1 11^1 = 660$.

10. $\gcd = 3 \cdot 5^2 \cdot 11 = 285$
$\text{lcm} = 2^4 \cdot 3^3 \cdot 5^3 \cdot 7^2 \cdot 11^2 \cdot 13 = 4,162,158,000$

11. $248396544 = 2^8 \cdot 3^6 \cdot 11^3$, so there are $(8+1)(6+1)(3+1) = 252$ possibilities for n.

12. Since $2a$ is a perfect square we have $a = 2x^2$, for some $x \in \mathbf{Z}^+$. Likewise, $3a$ a perfect cube $\Rightarrow a = 3^2y^3 = (3y)^2y$ for some $y \in \mathbf{Z}^+$. To minimize the value of a choose $y = 2$ and $x = 3y = 6$. Then $a = 72 = 2^3 \cdot 3^2$.

13. a) Proof: (i) Since $10|a^2$ we have $5|a^2$ and $2|a^2$. Then by Lemma 4.2 it follows that $5|a$ and $2|a$. So $a = 5b$ for some $b \in \mathbf{Z}^+$. Further, since $2|5b$ we have $2|5$ or $2|b$ (by Lemma 4.2). Consequently, $a = 5b = 5(2c) = 10c$, and 10 divides a.
(ii) This result is false — let $a = 2$.

b) We can generalize section (i) of part (a) by replacing 10 by an integer n of the form

$p_1p_2\cdots p_t$, a product of t distinct primes. (So n is a square-free integer — that is, no square greater than 1 divides n.)

14. Proof: We find that $abcabc = (abc)(1001) = (abc)(7)(11)(13)$.

15. Since $7! = 2^4 \cdot 3^2 \cdot 5 \cdot 7$, the smallest perfect square that is divisible by $7!$ is $2^4 \cdot 3^2 \cdot 5^2 \cdot 7^2 = (35) \times (7!) = 176,400$.

16. If $n \in \mathbf{Z}^+$ and n is a perfect square, then $n = p_1^{e_1}p_2^{e_2}\cdots p_k^{e_k}$, where p_i is prime and e_i is a positive even integer for all $1 \leq i \leq k$. Hence $(e_1+1)(e_2+1)\cdots(e_k+1)$ is a product of odd integers. Therefore the number of positive divisors of n is odd.
Conversely, if $n \in \mathbf{Z}^+$ and n is not a perfect square, then $n = p_1^{e_1}p_2^{e_2}\cdots p_k^{e_k}$ where each p_i is prime and e_i is odd for some $1 \leq i \leq k$. Therefore (e_i+1) is even for some $1 \leq i \leq k$, so $(e_1+1)(e_2+1)\cdots(e_k+1)$ is even and n has an even number of positive divisors.

17. For $1260 \times n$ to be a perfect cube, the exponent on each prime divisor must be a multiple of 3. Since $1260 = 2^2 \cdot 3^2 \cdot 5 \cdot 7$, we want $1260 \times n = 2^3 \cdot 3^3 \cdot 5^3 \cdot 7^3$, so $n = 2 \cdot 3 \cdot 5^2 \cdot 7^2 = 7350$.

18. (a) Since $200 = 2^3 \cdot 5^2$, the number of times the 200th coin will be turned over is $(4)(3) = 12$, the number of divisors of 200.
(b) The following coins will also be turned over 12 times:
(i) $2^3 \cdot 3^2 = 72$ (ii) $2^5 \cdot 3 = 96$ (iii) $2^2 \cdot 3^3 = 108$ (iv) $2^5 \cdot 5 = 160$
(c) The 192nd coin is turned over 14 times because $192 = 2^6 \cdot 3$.

19. (a) $4 = 2^2$; $8 = 2^3$; $16 = 2^4$; $32 = 2^5$.
Considering the powers of 2, there are 5 different sums of two distinct exponents: $5 = 2+3$; $6 = 2+4$; $7 = 2+5 = 3+4$; $8 = 3+5$; $9 = 4+5$. Hence there are 5 different products that we can form.
(b) Here we have 2^n for $n = 2, 3, 4, 5$ and 6. Now there are 7 different sums of two distinct exponents: $5 = 2+3$; $6 = 2+4$; $7 = 2+5 = 3+4$; $8 = 2+6 = 3+5$; $9 = 3+6 = 4+5$; $10 = 4+6$; $11 = 5+6$. Consequently, we can form 7 different products in this case.
(c) The set here may also be represented as $A \cup B$ where $A = \{2^n | n \in \mathbf{Z}^+, 2 \leq n \leq 6\}$ and $B = \{3^k | k \in \mathbf{Z}^+, 2 \leq k \leq 5\}$.
If the product uses two integers from A then there are 7 possibilities. If both integers are selected from B then we have 5 possibilities. Finally there are $5 \times 4 = 20$ products using one number from each of the sets A, B. In total, the number of different products is $7 + 5 + 20 = 32$.
(d) Consider the set given here as $A \cup B \cup C$ where $A = \{4, 8, 16, 32, 64\}$, $B = \{9, 27, 81, 243, 729\}$ and $C = \{25, 125, 625, 3125\}$.

Here there are six cases to enumerate.

(1) Both elements from A: 7 possibilities.
(2) Both elements from B: 7 possibilities.
(3) Both elements from C: 5 possibilities.
(4) One element from each of A, B: $5 \times 5 = 25$ possibilities.

(5) One element from each of A, C: $5 \times 4 = 20$ possibilities.
(6) One element from each of B, C: $5 \times 4 = 20$ possibilities.
In total there are $7 + 7 + 5 + 25 + 20 + 20 = 84$ possible products.
(e) This case generalizes the result in part (d). Once again there are 84 possible products.

20.

```
Program Primefactors (input,output);

Var
    p, j, k, n, originalvalue, count: integer;
Begin
Write ('The value of n is ');
    Read (n);
    originalvalue := n;
    Writeln ('The prime factorization of ', n:0, ' is ');
    If n Mod 2 = 0 Then
        Begin
            count := 0;
            While n Mod 2 = 0 Do
                Begin
                    count := count + 1;
                    n := n Div 2
                End;
            Write ('2(', count:0, ') ')
        End;
    If n Mod 3 = 0 Then
        Begin
            count := 0;
            While n Mod 3 = 0 Do
                Begin
                    count := count + 1;
                    n := n Div 3
                End;
            Write ('3(', count :0, ') ')
        End;
    p := 5;
    While n >= 5 Do
        Begin
            j := 1;
            Repeat
                j := j + 1;
                k := p Mod j
            Until (k = 0) Or (j = Trunc(Sqrt(p)));
            If (k <> 0) And (n Mod p = 0) Then
```

```
Begin
    count := 0;
        While n Mod p = 0 Do
            Begin
                count := count + 1;
                n := n Div p
            End;
        Write (p:0, ')', count:0, ') ')
    End;
  p := p + 2
 End;
End.
```

21. The length of $AB = 2^8 = 256$; the length of $AC = 2^9 = 512$. The perimeter of the triangle is 1061.

22. (a) $\prod_{i=1}^{10}(-1)^i = -1$

(b) $\prod_{i=1}^{2n+1}(-1)^i = (-1)^{(2n+1)(2n+2)/2} = (-1)^{(2n+1)(n+1)} = \begin{cases} 1, & \text{for } n \text{ odd} \\ -1, & \text{for } n \text{ even} \end{cases}$

(c) $\prod_{i=4}^{8}\frac{(i+1)(i+2)}{(i-1)(i)} = \left(\frac{5\cdot 6}{3\cdot 4}\right)\left(\frac{6\cdot 7}{4\cdot 5}\right)\left(\frac{7\cdot 8}{5\cdot 6}\right)\left(\frac{8\cdot 9}{6\cdot 7}\right)\left(\frac{9\cdot 10}{7\cdot 8}\right) = \frac{8\cdot 9\cdot 9\cdot 10}{3\cdot 4\cdot 4\cdot 5} = 27$

(d) $\prod_{i=n}^{2n}\frac{i}{2n-i+1} = \left(\frac{n}{n+1}\right)\left(\frac{n+1}{n}\right)\left(\frac{n+2}{n-1}\right)\cdots\left(\frac{2n-1}{2}\right)\left(\frac{2n}{1}\right) =$
$[(2n)!/(n-1)!]/(n+1)! = (2n)!/[(n-1)!(n+1)!] = \binom{2n}{n-1} = \binom{2n}{n+1}$

23. (a) From the Fundamental Theorem of Arithmetic $88,200 = 2^3 \cdot 3^2 \cdot 5^2 \cdot 7^2$. Consider the set $F = \{2^3, 3^2, 5^2, 7^2\}$. Each subset of F determines a factorization ab where $\gcd(a,b) = 1$. There are 2^4 subsets – hence, 2^4 factorizations. Since order is not relevant, this number (of factorizations) reduces to $(1/2)2^4 = 2^3$. And since $1 < a < n$, $1 < b < n$, we remove the case for the empty subset of F (or the subset F itself). This yields $2^3 - 1$ such factorizations.

(b) Here $n = 2^3 \cdot 3^2 \cdot 5^2 \cdot 7^2 \cdot 11$ and there are $2^4 - 1$ such factorizations.

(c) Suppose that $n = p_1^{n_1}p_2^{n_2}\cdots p_k^{n_k}$, where $p_1, p_2, \ldots, p_k$ are k distinct primes and $n_1, n_2, \ldots, n_k \geq 1$. The number of unordered factorizations of n as ab, where $1 < a < n$, $1 < b < n$, and $\gcd(a,b) = 1$, is $2^{k-1} - 1$.

24. (a) $\prod_{i=1}^{5}(i^2 + i)$ (b) $\prod_{i=1}^{5}(1 + x^i)$ (c) $\prod_{i=1}^{6}(1 + x^{2i-1})$

25. Proof: (By Mathematical Induction)
For $n = 2$ we find that $\prod_{i=2}^{2}(1 - \frac{1}{i^2}) = (1 - \frac{1}{2^2}) = (1 - \frac{1}{4}) = 3/4 = (2+1)/(2 \cdot 2)$, so the result is true in this first case and this establishes the basis step for our inductive proof. Next we assume the result true for some (particular) $k \in \mathbf{Z}^+$ where $k \geq 2$. This gives us $\prod_{i=2}^{k}(1 - \frac{1}{i^2}) = (k+1)/(2k)$. When we consider the case for $n = k+1$, using the inductive step, we find that
$$\prod_{i=2}^{k+1}(1 - \frac{1}{i^2})) = \left(\prod_{i=2}^{k}(1 - \frac{1}{i^2})\right)(1 - \frac{1}{(k+1)^2}) = [(k+1)/(2k)][1 - \frac{1}{(k+1)^2}] =$$
$$\left[\frac{k+1}{2k}\right]\left[\frac{(k+1)^2 - 1}{(k+1)^2}\right] = \frac{k^2 + 2k}{(2k)(k+1)} = (k+2)/(2(k+1)) = ((k+1)+1)/(2(k+1)).$$
The result now follows for all positive integers $n \geq 2$ by the Principle of Mathematical Induction.

26. (a) When n is a prime then it has exactly two positive divisors — namely, 1 and n.
(b) If $n = p^2$, where p is a prime, then n has exactly three positive divisors — namely, 1, p, and p^2.
(c) Let p, q denote two distinct primes. If $n = p^3$ or $n = pq$, then n has exactly four positive divisors — $1, p, p^2$, and p^3 for $n = p^3$, and $1, p, q$ and pq for $n = pq$.
(d) If $n = p^4$, where p is a prime, then n has exactly five positive divisors — namely, $1, p, p^2, p^3$, and p^4.

27. (a) The positive divisors of 28 are 1, 2, 4, 7, 14, and 28, and $1 + 2 + 4 + 7 + 14 + 28 = 56 = 2(28)$, so 28 is a perfect integer.
The positive divisors of 496 are 1, 2, 4, 8, 16, 31, 62, 124, 248, and 496, and $1 + 2 + 4 + 8 + 16 + 31 + 62 + 124 + 248 + 496 = 992 = 2(496)$, so 496 is a perfect integer.
(b) It follows from the Fundamental Theorem of Arithmetic that the divisors of $2^{m-1}(2^m - 1)$, for $2^m - 1$ prime, are $1, 2, 2^2, 2^3, \ldots, 2^{m-1}$, and $(2^m - 1)$, $2(2^m - 1)$, $2^2(2^m - 1)$, $2^3(2^m - 1), \ldots$, and $2^{m-1}(2^m - 1)$.
These divisors sum to $[1 + 2 + 2^2 + 2^3 + \ldots + 2^{m-1}] + (2^m - 1)[1 + 2 + 2^2 + 2^3 + \ldots + 2^{m-1}] = (2^m - 1) + (2^m - 1)(2^m - 1) = (2^m - 1)[1 + (2^m - 1)] = 2^m(2^m - 1) = 2[2^{m-1}(2^m - 1)]$, so $2^{m-1}(2^m - 1)$ is a perfect integer.

Supplementary Exercises

1. $a + (a + d) + (a + 2d) + \ldots + (a + (n-1)d) = na + [(n-1)nd]/2$. For $n = 1, a = a + 0$, and the result is true in this case. Assuming that $\sum_{i=1}^{k}[a + (i-1)d] = ka + [(k-1)kd]/2$, we have $\sum_{i=1}^{k+1}[a + (i-1)d] = (ka + [(k-1)kd]/2) + (a + kd) = (k+1)a + [k(k+1)d]/2$, so the result follows for all $n \in \mathbf{Z}^+$ by the Principle of Mathematical Induction.

2. Let t be the number of times the **while** loop is executed. Then we have
$sum = 10+17+24+\ldots+(10+7(t-1)) = 10+(10+7)+(10+14)+\ldots+(10+7(t-1))$.

From the previous exercise we know that for each $t \in \mathbf{Z}^+$
$a+(a+d)+(a+2d)+\ldots+(a+(t-1)d) = ta+[(t-1)(t)d]/2$, so here we have
$$sum = 10t + (7/2)t(t-1).$$
For $t = 52$, we have sum $= 9802$, and for $t = 53$, we find sum $= 10176$. Therefore, n, the last summand, is $10 + 7(52-1) = 367$.

3. Conjecture: $\sum_{i=1}^{n}(-1)^{i+1}i^2 = (-1)^{n+1}\sum_{i+1}^{n} i$, for all $n \in \mathbf{Z}^+$.
Proof: (By the Principle of Mathematical Induction)
If $n = 1$ the conjecture provides $\sum_{i=1}^{1}(-1)^{i+1}i^2 = (-1)^{1+1}(1)^2 = 1 = (-1)^{1+1}(1) = (-1)^{1+1}\sum_{i=1}^{1} i$, which is a true statement. This establishes the basis step of the proof.
In order to confirm the inductive step, we shall assume the truth of the result

$$\sum_{i=1}^{k}(-1)^{i+1}i^2 = (-1)^{k+1}\sum_{i=1}^{k} i$$

for some (particular) $k \geq 1$. When $n = k+1$ we find that

$$\sum_{i=1}^{k+1}(-1)^{i+1}i^2 = (\sum_{i=1}^{k}(-1)^{i+1}i^2) + (-1)^{(k+1)+1}(k+1)^2$$
$$= (-1)^{k+1}\sum_{i=1}^{k} i + (-1)^{k+2}(k+1)^2 = (-1)^{k+1}(k)(k+1)/2 + (-1)^{k+2}(k+1)^2$$
$$= (-1)^{k+2}[(k+1)^2 - (k)(k+1)/2]$$
$$= (-1)^{k+2}(1/2)[2(k+1)^2 - k(k+1)]$$
$$= (-1)^{k+2}(1/2)[2k^2 + 4k + 2 - k^2 - k]$$
$$= (-1)^{k+2}(1/2)[k^2 + 3k + 2] = (-1)^{k+2}(1/2)(k+1)(k+2)$$

$= (-1)^{k+2}\sum_{i=1}^{k+1} i$, so the truth of the result at $n = k$ implies the truth at $n = k+1$ — and we have the inductive step.
It then follows by the Principle of Mathematical Induction that

$$\sum_{i=1}^{n}(-1)^{i+1}i^2 = (-1)^{n+1}\sum_{i=1}^{n} i,$$

for all $n \in \mathbf{Z}^+$.

4. (a) $S(n) : 5|(n^5 - n)$. For $n = 1, n^5 - n = 0$ and $5|0$, so $S(1)$ is true. Assume $S(k)$: $5|(k^5 - k)$. For $n = k+1, (k+1)^5 - (k+1) = (k^5 - k) + 5k^4 + 10k^3 + 10k^2 + 5k$. Based on $S(k), 5|((k+1)^2 - (k+1))$ so $S(k) \Longrightarrow S(k+1)$ and the result is true for all $n \in \mathbf{Z}^+$ by the Principle of Mathematical Induction.

(b) $S(n) : 6|(n^3 + 5n)$. When $n = 1, n^3 + 5n = 6$, so $S(1)$ is true. Assuming $S(k)$, consider $S(k+1)$. $(k+1)^3 + 5(k+1) = (k^3 + 5k) + 6 + 3(k)(k+1)$. Since one of $k, k+1$ must be even, $6|[3(k)(k+1)]$. Then assuming $S(k)$ we have $6|[(k+1)^3+5(k+1)]$ and the general result is true for all $n \in \mathbf{Z}^+$ by the Principle of Mathematical Induction.

5. (a)

n	n^2+n+41	n	n^2+n+41	n	n^2+n+41
1	43	4	61	7	97
2	47	5	71	8	113
3	53	6	83	9	131

(b) For $n = 39, n^2+n+41 = 1601$, a prime. But for $n = 40, n^2+n+41 = (41)^2$, so $S(39) \not\Longrightarrow S(40)$.

6. (b) $s_4 = 119/120 = (5!-1)/5!$ $\quad s_5 = 719/720 = (6!-1)/6!$
$s_6 = 5039/5040 = (7!-1)/7!$

(c) $s_n = [(n+1)!-1]/(n+1)!$

(d) Based on the calculation in part (a) the conjecture is true for $n = 1$. Assuming that $s_k = [(k+1)!-1]/(k+1)!$, for $k \in \mathbf{Z}^+$, consider s_{k+1}.
$s_{k+1} = s_k + (k+1)/(k+2)! = [(k+1)!-1]/(k+1)! + (k+1)/(k+2)! = [(k+2)!-(k+2)+(k+1)]/(k+2)! = [(k+2)!-1]/(k+2)!$, so the result follows for all $n \in \mathbf{Z}^+$ by the Principle of Mathematical Induction.

7. (a) For $n = 0, 2^{2n+1}+1 = 2+1 = 3$, so the result is true in this first case. Assuming that 3 divides $2^{2k+1}+1$ for $n = k \in \mathbf{N}$, consider the case of $n = k+1$. Since $2^{2(k+1)+1}+1 = 2^{2k+3}+1 = 4(2^{2k+1})+1 = 4(2^{2k+1}+1)-3$, and 3 divides both $2^{2k+1}+1$ and 3, it follows that 3 divides $2^{2(k+1)+1}+1$. Consequently, the result is true for $n = k+1$ whenever it is true for $n = k$. So by the Principle of Mathematical Induction the result follows for all $n \in \mathbf{N}$.

(b) When $n = 0, 0^3+(0+1)^3+(0+2)^3 = 9$, so the statement is true in this case. We assume the truth of the result when $n = k \geq 0$ and examine the result for $n = k+1$. We find that $(k+1)^3+(k+2)^3+(k+3)^3 = (k+1)^3+(k+2)^3+[k^3+9k^2+27k+27] = [k^3+(k+1)^3+(k+2)^3]+[9(k^2+3k+3)]$, where the first summand is divisible by 9 because of the induction hypothesis. Consequently, since the result is true for $n = 0$, and since the truth at $n = k$ (≥ 0) implies the truth for $n = k+1$, it follows from the Principle of Mathematical Induction that the statement is true for all integers $n \geq 0$.

8. Proof: There are four cases to consider.
(1) $n = 10m+1$. Here n^1 satisfies the condition sought.
(2) $n = 10m+3$. Here $n^2 = 100m^2+60m+9$ and $n^4 = 10000m^4+12000m^3+5400m^2+1080m+81 = 10(1000m^4+1200m^3+540m^2+108m+8)+1$, so the units digit of n^4 is 1.
(3) $n = 10m+7$. As in case (2) we need n^4. For $n^2 = 100m^2+140m+49$, and $n^4 = 10000m^4+28000m^3+29400m^2+13720m+2401 = 10(100m^4+2800m^3+2940m^2+1372m+240)+1$, where the units digit is 1.
(4) $n = 10m+9$. Fortunately we only need n^2 here, since $n^2 = 100m^2+180m+81 = 10(10m^2+18m+8)+1$.
[Note: For any $n \in \mathbf{Z}^+$, where n is odd and *not* divisible by 5, we always find the units digit in n^4 to be 1.]

9. Converting to base 10 we find that $81x+9y+z=36z+6y+x$, and so $80x+3y-35z=0$. Since $5|(80x-35z)$ and gcd $(3,5)=1$, it follows that $5|y$. Consequently, $y=0$ or $y=5$.

For $y=0$ the equation $80x-35z=0$ leads us to $16x-7z=0$ and $16x=7z \Rightarrow 16|z$. Since $0 \leq z \leq 5$ we find here that $z=0$ and the solution is $x=y=z=0$.

If $y=5$, then $80x+15-35z=0 \Rightarrow 16x+3-7z=0$. With $0 \leq x, z \leq 5$, $16x=7z-3 \Rightarrow z$ is odd and $z=1,3$, or 5. Since 16 does *not* divide $4(=7(1)-3)$ or $18(=7(3)-3)$, and since 16 *does* divide $32(=7(5)-3)$ we find that $z=5$ and $x=2$. Hence $x=2$, $y=5$, and $z=5$. [And we see that $(xyz)_9=81x+9y+z=81(2)+9(5)+5=212=36(5)+6(5)+2=36z+6y+x=(zyx)_6$.]

10. From the Fundamental Theorem of Arithmetic we have $3000=2^3\cdot 3^1\cdot 5^3$, so 3000 has $(3+1)(1+1)(3+1)=32$ divisors. Since $\gcd(n,n+3000)$ is a divisor of 3000, there are 32 possibilities – depending on the value of n.

11. For $n=2$ we find that $2^2=4<6=\binom{4}{2}<16=4^2$, so the statement is true in this first case.

Assuming the result true for $n=k\geq 2$ – i.e., $2^k<\binom{2k}{k}<4^k$, we now consider what happens for $n=k+1$. Here we find that

$\binom{2(k+1)}{k+1}=\binom{2k+2}{k+1}=\left[\frac{(2k+2)(2k+1)}{(k+1)(k+1)}\right]\binom{2k}{k}=2[(2k+1)/(k+1)]\binom{2k}{k}>2[(2k+1)/(k+1)]2^k>2^{k+1}$, since $(2k+1)/(k+1)=[(k+1)+k]/(k+1)>1$. In addition, $[(k+1)+k)]/(k+1)<2$,

so $\binom{2k+2}{k+1}=2[(2k+1)/(k+1)]\binom{2k}{k}<(2)(2)\binom{2k}{k}<4^{k+1}$. Consequently the result is true for all $n\geq 2$ by the Principle of Mathematical Induction.

12. For $n=1, 7^3+8^3=855=(57)(15)$. Assuming that $57|(7^{k+2}+8^{2k+1})$, since $7^{(k+1)+2}+8^{2(k+1)+1}=7^{k+3}+8^{2k+3}=7(7^{k+2})+64(8^{2k+1})=64(7^{k+2})+64(8^{2k+1})-57(7^{k+2})$, we have $57|(7^{k+3}+8^{2k+3})$, so the result follows by the Principle of Mathematical Induction.

13. First we observe that the statement is true for all $n\in\mathbf{Z}^+$ where $64\leq n\leq 68$. This follows from the calculations:

$64=2(17)+6(5)$ $\quad 65=13(5)$ $\quad 66=3(17)+3(5)$ $\quad 67=1(17)+10(5)$ $\quad 68=4(17)$

Now assume the result is true for all n where $68\leq n\leq k$ and consider the integer $k+1$. Then $k+1=(k-4)+5$, and since $64\leq k-4<k$ we can write $k-4=a(17)+b(5)$ for some $a,b\in\mathbf{N}$. Consequently, $k+1=a(17)+(b+1)(5)$, and the result follows for all $n\geq 64$ by the Alternative Form of the Principle of Mathematical Induction.

14. To find all such a,b we solve the Diophantine equation $12a+7b=1$. Since $\gcd(12,7)=1$, we start with the Euclidean algorithm:

$$\begin{aligned} 12 &= 1\cdot 7+5, & 0<5<7 \\ 7 &= 1\cdot 5+2, & 0<2<5 \\ 5 &= 2\cdot 2+1, & 0<1<2 \\ 2 &= 2\cdot 1 & \end{aligned}$$

Then $1 = 5 - 2 \cdot 2 = 5 - 2[7-5] = (-2)7 + 3(5) = (-2)7 + 3(12-7) = 12 \cdot 3 + 7(-5) = 12[3+7k] + 7[-5-12k], k \in \mathbf{Z}$. Hence

$$a = 3 + 7k, \quad b = -5 - 12k, \quad k \in \mathbf{Z}.$$

15. (a) $r = r_0 + r_1 \cdot 10 + r_2 \cdot 10^2 + \ldots + r_n \cdot 10^n = r_0 + r_1(9) + r_1 + r_2(99) + r_2 + \ldots + r_n \underbrace{(99\ldots9)}_{n\ 9\text{'s}} + r_n = [9r_1 + 99r_2 + \ldots + (99\ldots9)r_n] + (r_0 + r_1 + r_2 + \ldots + r_n)$. Hence $9|r$ iff $9|(r_0 + r_1 + r_2 + \ldots + r_n)$.
(c) $3|t$ for $x = 1$ or 4 or 7; $9|t$ for $x = 7$.

16. $50x + 20y = 620 \Longrightarrow 5x + 2y = 62$
$\gcd(5,2) = 1$ and $1 = 5(1) + 2(-2)$ so $62 = 5(62) + 2(-124) = 5(62 - 2k) + 2(-124 + 5k)$, $k \in \mathbf{Z}$. $x = 62 - 2k \geq 0 \Longrightarrow 31 \geq k$; $y = -124 + 5k \geq 0 \Longrightarrow k \geq 24.8$
Solutions: (1) $k = 25 : x = 12, y = 1$; (2)$k = 26 : x = 10, y = 6$; (3)$k = 27 : x = 8, y = 11$; (4)$k = 28 : x = 6, y = 16$: (5)$k = 29 : x = 4, y = 21$; (6)$k = 30 : x = 2, y = 26$; (7)$k = 31 : x = 0, y = 31$.

17. (a) Let $n = 2^{e_1} \cdot 3^{e_2} \cdot 5^{e_3} \cdot 7^{e_4} \cdot 11^{e_5}$ where $e_1 + e_2 + e_3 + e_4 + e_5 = 9$, with $e_i \geq 0$ for all $1 \leq i \leq 5$. The number of solutions to this equation is $\binom{5+9-1}{9} = \binom{13}{9}$
(b) $\binom{8}{4}$

18. (a) $2^{4(1+2+3)}5^{4(1+2+3)}$ (b) $2^{5(1+2+3)}\ 5^{4(1+2+3+4)}$
(c) $2^{2(4)(1+2+3)}5^{2(4)(1+2+3)}7^{4(4)(1)}$ (d) $2^{3(4)(1+2+3)}3^{4(4)(1+2)}5^{3(4)(1+2+3)}$
(e) $p^e q^f$, where $e = (n+1)(1 + 2 + \cdots + m) = (n+1)(m)(m+1)/2$ and $f = (m+1)(1 + 2 + \cdots + n) = (m+1)(n)(n+1)/2$
(f) $p^e q^f r^g$, when $e = (n+1)(k+1)(1 + 2 + \cdots + m) = (n+1)(k+1)(m)(m+1)/2$, $f = (m+1)(k+1)(1 + 2 + \cdots + n) = (m+1)(k+1)(n)(n+1)/2$, and $g = (m+1)(n+1)(1 + 2 + \cdots + k) = (m+1)(n+1)(k)(k+1)/2$.

19. (a) 1,4,9.
(b) 1,4,9,16,..., k where k is the largest square less than or equal to n.

20. Proof: For $1 \leq i \leq 5$, it follows from the division algorithm that $a_i = 5q_i + r_i$, where $0 \leq r_i \leq 4$. So now we shall consider the remainders: r_1, r_2, r_3, r_4, r_5. For if a selection of the remainders adds to a multiple of 5, then the sum of the corresponding elements of A will also sum to a multiple of 5. (Note that for the remainders we need not have five distinct integers.)
1) If $r_i = 0$ for some $1 \leq i \leq 5$, then $5|a_i$ and we are finished. Therefore we shall assume from this point on that $r_i \neq 0$ for all $1 \leq i \leq 5$.
2) If $1 \leq r_1 = r_2 = r_3 = r_4 = r_5 \leq 4$, then $a_1 + a_2 + \ldots + a_5 = 5(q_1 + q_2 + \ldots q_5) + 5r_1$, and the result follows. Consequently we now narrow our attention to the cases where at least two different nonzero remainders occur.

Case 1: (There are at least three 4's). Here the possibilities to consider are (i) $4 + 1$; (ii) $4 + 4 + 2$; and (iii) $4 + 4 + 4 + 3$ — these all lead to the result we are seeking.

Case 2: (We have one or two 4's). If there is at least one 1, or at least one 2 and one 3, then we are done. Otherwise we get one of the following possibilities: (i) 4 + 2 + 2 + 2 or (ii) 4 + 3 + 3.

Case 3: (Now there are no 4's and at least one 3.) Then we either have (i) 3 + 2; (ii) 3 + 1 + 1; or (iii) 3 + 3 + 3 + 1.

Case 4: (We now have only 1's and 2's as summands). The final possibilities are (i) 2 + 1 + 1 + 1; and (ii) 2 + 2 + 1.

21. (a) For all $n \in \mathbf{Z}^+$, $n \geq 3$, $1+2+3+\ldots+n = n(n+1)/2$. If $\{1,2,3,\ldots,n\} = A \cup B$ with $s_A = s_B$, then $2s_A = n(n+1)/2$, or $4s_A = n(n+1)$. Since $4|n(n+1)$ and $\gcd(n, n+1) = 1$ then either $4|n$ or $4|(n+1)$.

(b) Here we are verifying the converse of our result in part (a).
(i) If $4|n$ we write $n = 4k$. Here we have $\{1,2,3,\ldots,k,k+1,\ldots,3k,3k+1,\ldots,4k\} = A \cup B$ where $A = \{1,2,3,\ldots,k,3k+1,3k+2,\ldots,4k-1,4k\}$ and $B = \{k+1,k+2,\ldots,2k,2k+1,3k-1,3k\}$, with $s_A = (1+2+3+\ldots+k) + [(3k+1)+(3k+2)+\ldots+(3k+k)] = [k(k+1)/2] + k(3k) + [k(k+1)/2] = k(k+1) + 3k^2 = 4k^2+k$, and $s_B = [(k+1)+(k+2)+\ldots+(k+k)] + [(2k+1)+(2k+2)+\ldots+(2k+k)] = k(k) + [k(k+1)/2] + k(2k) + [k(k+1)/2] = 3k^2 + k(k+1) = 4k^2+k$.
(ii) Now we consider the case where $n+1 = 4k$. Then $n = 4k-1$ and we have $\{1,2,3,\ldots,k-1,k,\ldots,3k-1,3k,\ldots,4k-2,4k-1\} = A \cup B$, with $A = \{1,2,3,\ldots,k-1,3k,3k+1,\ldots,4k-1\}$ and $B = \{k,k+1,\ldots,2k-1,2k,2k+1,\ldots,3k-1\}$. Here we find $s_A = [1+2+3+\ldots+(k-1)] + [3k+(3k+1)+\ldots+(3k+(k-1))] = [(k-1)(k)/2] + k(3k) + [(k-1)(k)/2] = 3k^2+k^2-k = 4k^2-k$, and $s_B = [k+(k+1)+\ldots+(k+(k-1))] + [2k+(2k+1)+\ldots+(2k+(k-1))] = k^2 + [(k-1)(k)/2] + k(2k) + [(k-1)(k)/2] = 3k^2+(k-1)k = 4k^2-k$.

22. Let n be one such integer. Then $5n-4 = 6s$ and $7n+1 = 4t$, for some $s, t \in \mathbf{Z}$. Since $2|4$ and $2|6$, it follows that $2|5n$ because $5n-4 = 6s$. From Lemma 4.2 we have $2|n$. Consequently, as $7n+1 = 4t$, we find that $2|1$. This contradiction tells us that no such integer n exists.

23. (a) The result is true for $a = 1$, so consider $a > 1$. From the Fundamental Theorem of Arithmetic we can write $a = p_1^{e_1} p_2^{e_2} \cdots p_t^{e_t}$, where $p_1, p_2, \ldots, p_t$, are t distinct primes and $e_i > 0$, for all $1 \leq i \leq t$. Since $a^2|b^2$ it follows that $p_i^{2e_i}|b^2$ for all $1 \leq i \leq t$. So $b^2 = p_1^{2f_1} p_2^{2f_2} \cdots p_t^{2f_t} c^2$, where $f_i \geq e_i$ for all $1 \leq i \leq t$, and $b = p_1^{f_1} p_2^{f_2} \cdots p_t^{f_t} c = a(p_1^{f_1-e_1} p_2^{f_2-e_2} \cdots p_t^{f_t-e_t})c$, where $f_i - e_i \geq 0$ for all $\leq i \leq t$. Consequently, $a|b$.

(b) This result is not necessarily true! Let $a = 8$ and $b = 4$. Then $a^2 (= 64)$ divides $b^3 (= 4)$, but a does *not* divide b.

24. Proof: Suppose that $n > 1$. If n is not prime, then $n = n_1 n_2$ where $1 < n_1 < n$ and $1 < n_2 < n$. Since $n|n$ we have $n|n_1 n_2$. So $n|n_1$ or $n|n_2$ — where either result is impossible.

25. (a) Recall that

$$\begin{aligned} a^3+b^3 &= (a+b)(a^2-ab+b^2)\\ a^5+b^5 &= (a+b)(a^4-a^3b+a^2b^2-ab^3+b^4)\\ &\vdots\\ a^p+b^p &= (a+b)(a^{p-1}-a^{p-2}b+\cdots+b^{p-1})\\ &= (a+b)\sum_{i=1}^{p} a^{p-i}(-b)^{i-1}, \end{aligned}$$

for p an odd prime.

Since k is not a power of 2 we write $k = r\cdot p$, where p is an odd prime and $r \geq 1$. Then $a^k + b^k = (a^r)^p + (b^r)^p = (a^r + b^r)\sum_{i=1}^{p} a^{r(p-i)}(-b)^{r(i-1)}$, so $a^k + b^k$ is composite.

(b) Here n is not a power of 2. If, in addition, n is not prime, then $n = r\cdot p$ where p is an odd prime. Then $2^n+1 = 2^n+1^n = 2^{r\cdot p}+1^{r\cdot p} = (2^r+1^r)\sum_{i=1}^{p} 2^{r(p-i)}(-1)^{r(i-1)} = (2^r+1)\sum_{i=1}^{p} 2^{r(p-i)}$, so 2^n+1 is composite — not prime.

26. Proof: Here the open statement $S(n)$ represents: $H_{2^n} \leq 1+n$, and for the basis step we consider what happens at $n=0$. We find that $H_{2^n} = H_{2^0} = H_1 = 1 \leq 1+0 = 1+n$, so $S(n)$ is true for this first case (where $n=0$).

Assuming the truth of $S(k)$ for some k in $\mathbf{N}$ (not just $\mathbf{Z}^+$), we obtain the induction hypothesis

$$S(k): \quad H_{2^k} \leq 1+k.$$

Continuing with the inductive step we now examine $S(n)$ for $n = k+1$. We find that

$$\begin{aligned} H_{2^{k+1}} &= [1+\tfrac{1}{2}+\tfrac{1}{3}+\cdots+\tfrac{1}{2^k}] + [\tfrac{1}{(2^k+1)}+\tfrac{1}{(2^k+2)}+\cdots+\tfrac{1}{(2^k+2^k)}]\\ &= H_{2^k} + [\tfrac{1}{(2^k+1)}+\tfrac{1}{(2^k+2)}+\cdots+\tfrac{1}{(2^k+2^k)}]. \end{aligned}$$

Since $\frac{1}{(2^k+j)} < \frac{1}{2^k}$, for all $1 \leq j \leq 2^k$, it follows that

$$H_{2^{k+1}} \leq H_{2^k} + (2^k)(\frac{1}{2^k}) = H_{2^k} + 1.$$

And now from the induction hypothesis we deduce that

$$H_{2^{k+1}} \leq H_{2^k} + 1 \leq (1+k)+1 = 1+(k+1),$$

so the result $S(n)$ is true for all $n \in \mathbf{N}$ — by virtue of the Principle of Mathematical Induction.

27. Proof: For $n=0$ we find that $F_0 = 0 \leq 1 = (5/3)^0$, and for $n=1$ we have $F_1 = 1 \leq (5/3) = (5/3)^1$. Consequently, the given property is true in these first two cases (and this provides the basis step of the proof).

Assuming that this property is true for $n = 0,1,2,\ldots,k-1,k$, where $k \geq 1$, we now examine what happens at $n = k+1$. Here we find that

$F_{k+1} = F_k + F_{k-1} \le (5/3)^k + (5/3)^{k-1} = (5/3)^{k-1}[(5/3)+1] = (5/3)^{k-1}(8/3)$
$= (5/3)^{k-1}(24/9) \le (5/3)^{k-1}(25/9) = (5/3)^{k-1}(5/3)^2 = (5/3)^{k+1}$.
It then follows from the Alternative Form of the Principle of Mathematical Induction that $F_n \le (5/3)^n$ for all $n \in \mathbf{N}$.

28. Proof: When $n = 0$ we find that

$$L_0 = \sum_{i=0}^{0} L_i = 2 = 3 - 1 = L_2 - 1 = L_{0+2} - 1,$$

so the claim is established in this first case.

For some $k \in \mathbf{N}$, where $k \ge 0$, now we assume true that

$$L_0 + L_1 + L_2 + \cdots + L_k = \sum_{i=0}^{k} L_i = L_{k+2} - 1.$$

Then for $n = k + 1 (\ge 1)$ we have

$$(*) \quad \sum_{i=0}^{k+1} L_i = \sum_{i=0}^{k} L_i + L_{k+1} = (L_{k+2} - 1) + L_{k+1} = (L_{k+2} + L_{k+1}) - 1 = L_{k+3} - 1 = L_{(k+1)+2} - 1,$$

and so we see how the truth at $n = k$ implies that at $k + 1$. Consequently, the summation formula is valid for all $n \in \mathbf{N}$ by the Principle of Mathematical Induction.
[Note that for the equations at (*), the first equality follows from the generalized associative law of addition — and the fourth equality rests upon the given recursive definition of the Lucas numbers since $k + 3 \ge 3 (\ge 2)$.]

29. a) There are $9 \cdot 10 \cdot 10 = 900$ such palindromes and their sum is $\sum_{a=1}^{9} \sum_{b=0}^{9} \sum_{c=0}^{9} abcba =$ $\sum_{a=1}^{9} \sum_{b=0}^{9} \sum_{c=0}^{9} (10001a + 1010b + 100c) = \sum_{a=1}^{9} \sum_{b=0}^{9} [10(10001a + 1010b) + 100(9 \cdot 10/2)] =$ $\sum_{a=1}^{9} \sum_{b=0}^{9} (100010a + 10100b + 4500) = \sum_{a=1}^{9} [10(100010a) + 10100(9 \cdot 10/2) + 10(4500)] =$ $1000100 \sum_{a=1}^{9} a + 9(454500) + 9(45000) = 1000100(9 \cdot 10/2) + 4090500 + 405000 = 49,500,000.$

b)
```
begin
    sum := 0
    for a := 1 to 9 do
        for b := 0 to 9 do
            for c := 0 to 9 do
                sum := sum + 10001 * a + 1010 * b + 100 * c
    print sum
end
```

30. Proof: Let $c = \gcd(a,b)$, $d = \gcd(\frac{a-b}{2}, b)$.
Since a, b are odd, it follows that $a - b$ is even and $a - b = 2(\frac{a-b}{2})$, with $\frac{a-b}{2} \in \mathbf{Z}^+$. Also, c is odd since a, b are odd. Now $c = \gcd(a,b) \Rightarrow c \mid a$ and $c \mid b \Rightarrow c \mid (a-b) \Rightarrow c \mid (\frac{a-b}{2})$ because $\gcd(2,c) = 1$. Consequently, $c \mid b$ and $c \mid (\frac{a-b}{2}) \Rightarrow c \mid d$.

As $d = \gcd(\frac{a-b}{2}, b)$, it follows that $d \mid 2(\frac{a-b}{2}) + b$, that is, $d \mid a$. Since $d \mid a$ and $d \mid b$, we have $d \mid c$.

Since $c \mid d$ and $d \mid c$ and $c, d > 0$, it follows that $c = d$.

31. Proof: Suppose that $7|n$. We see that $7|n \Rightarrow 7|(n-21u) \Rightarrow 7|[(n-u)-20u] \Rightarrow 7|[10(\frac{n-u}{10})-20u] \Rightarrow 7|[10(\frac{n-u}{10} - 2u)] \Rightarrow 7|(\frac{n-u}{10} - 2u)$, by Lemma 4.2 since $\gcd(7,10) = 1$. [Note: $\frac{n-u}{10} \in \mathbf{Z}^+$ since the units digit of $n - u$ is 0.] Conversely, if $7|(\frac{n-u}{10} - 2u)$, then since $\frac{n-u}{10} - 2u = \frac{n-21u}{10}$ we find that $7|(\frac{n-21u}{10}) \Rightarrow 7 \cdot 10 \cdot x = n - 21u$, for some $x \in \mathbf{Z}^+$. Since $7|7$ and $7|21$, it then follows that $7|n$ – by part (e) of Theorem 4.3.

32. a) If $19m + 90 + 8n = 1998$, then $m = (1/19)(1908 - 8n)$. Since $1908 = 19(100) + 8$, the remainder for $8n/19$ must be 8. This occurs for $n = 1$, and then $m = (1/19)(1908 - 8) = (1/19)(1900) = 100$.
b) In a similar way we have $n = (1/8)(1908 - 19m)$. Here $1908 = 8(238) + 4$, so the remainder for $19m/8$ must be 4. This occurs for $m = 4$ (and *not* for $m = 1, 2$, or 3), and then $n = (1/8)(1908 - 76) = 229$.

33. If Catrina's selection includes any of 0,2,4,6,8, then at least two of the resulting three-digit integers will have an even unit's digit, and be even – hence, *not* prime. Should her selection include 5, then two of the resulting three-digit integers will have 5 as their unit's digit; these three-digit integers are then divisible by 5 and so, they are *not* prime. Consequently, to complete the proof we need to consider the four selections of size 3 that Catrina can make from $\{1,3,7,9\}$. The following provides the selections – each with a three-digit integer that is not prime.
(1) $\{1,3,7\} : 713 = 23 \cdot 31$
(2) $\{1,3,9\} : 913 = 11 \cdot 83$
(3) $\{1,7,9\} : 917 = 7 \cdot 131$
(4) $\{3,7,9\} : 793 = 13 \cdot 61$

34. Let $T = \{a,b,c,d,e,f,g,h\}$ represent the eight element subset of $\{2,3,4,7,10,11,12,13,15\}$ that we use.

a	b	14	c
d	5	e	9
1	f	g	h

These numbers are placed in the table as shown in the figure. Since each row has the same average, it follows that $\frac{a+b+14+c}{4} = \frac{d+5+e+9}{4} = \frac{1+f+g+h}{4}$. Likewise, from the columns of the table we learn that $\frac{a+d+1}{3} = \frac{b+5+f}{3} = \frac{14+e+g}{3} = \frac{c+9+h}{3}$. Consequently, both 3 and 4 divide $s = (a+b+c+d+e+f+g+h+29)$, and since $\gcd(3,4) = 1$ it follows that 12 divides s. So we may write $s = 12k$. The total of the nine given integers is 77. If we let i denote the integer not in T, then $s = (77 - i) + 29$ so $77 - i + 29 = 12k$, or $106 - i = 12k$.

As we examine the nine given integers we see that

$106 - 2 = 104 = 12(8) + 8$	$106 - 7 = 99 = 12(8) + 3$	$106 - 12 = 94 = 12(7) + 10$
$106 - 3 = 103 = 12(8) + 7$	$106 - 10 = 96 = 12(8)$	$106 - 13 = 93 = 12(7) + 9$
$106 - 4 = 102 = 12(8) + 6$	$106 - 11 = 95 = 12(7) + 11$	$106 - 15 = 91 = 12(7) + 7$

Therefore we do not place 10 in the table. So $T = \{2, 3, 4, 7, 11, 12, 13, 15\}$ and the 12 entries in the table total $67 + 29 = 96$. It then follows that $a + b + 14 + c = d + 5 + e + 9 = 1 + f + g + h = 32$ and $a + d + 1 = b + 5 + f = 14 + e + g = c + 9 + h = 24$.

From column 3 we have $14 + e + g = 24$, so $e + g = 10$. The entries in T imply that $\{e, g\} = \{3, 7\}$; $e = 3 \Rightarrow d = 15$ (from the equation $d + 5 + e + 9 = 32$). With $d = 15$, from $a + d + 1 = 24$ we have $a = 8$, but $8 \notin T$. Consequently, $e = 7$ and $g = 3$, and then $d = 32 - 21 = 11$. Column 1 indicates that $a + d + 1 = 24$ so $a = 12$. From column 2 it follows that $b + f = 19$, so $\{b, f\} = \{4, 15\}$. As $b = 15 \Rightarrow a + b + 14 + c > 32$, it follows that $b = 4$ and $f = 15$. Row 1 then indicates that $c = 32 - a - b - 14 = 2$ and from row 3 (or column 4) we deduce that $h = 13$. The completed table is shown in the figure.

12	4	14	2
11	5	7	9
1	15	3	13

35. Let x denote the integer Barbara erased. The sum of the integers $1, 2, 3, \ldots, x - 1, x + 1, x + 2, \ldots, n$ is $[n(n+1)/2] - x$, so $[[n(n+1)/2] - x]/(n - 1) = 35\frac{7}{17}$. Consequently, $[n(n+1)/2] - x = (35\frac{7}{17})(n - 1) = (602/17)(n - 1)$. Since $[n(n+1)/2] - x \in \mathbf{Z}^+$, it follows that $(602/17)(n - 1) \in \mathbf{Z}^+$. Therefore, from Lemma 4.2, we find that $17|(n - 1)$ because 17 does not divide 602. For $n = 1, 18, 35, 52$ we have:

n	$x = [n(n+1)/2] - (602/17)(n-1)$
1	1
18	-431
35	-574
52	-428

When $n = 69$, we find that $x = 7$ [and $(\sum_{i=1}^{69} i - 7)/68 = 602/17 = 35\frac{7}{17}$].

For $n = 69 + 17k$, $k \geq 1$, we have

$$\begin{aligned} x &= [(69 + 17k)(70 + 17k)/2] - (602/17)[68 + 17k] \\ &= 7 + (k/2)[1159 + 289k] \\ &= [7 + (1159k/2)] + (289k^2)/2 > n. \end{aligned}$$

Hence the answer is unique: namely, $n = 69$ and $x = 7$.

36. Let $S = \{1, 2, 3, \ldots, 100\}$ be the sample space for this experiment and let A, B, C denote the following events:

A: Leslie's selection is divisible by 2: $\{2, 4, 6, \ldots, 98, 100\}$

B: Leslie's selection is divisible by 3: $\{3, 6, 9, \ldots, 96, 99\}$

C: Leslie's selection is divisible by 5: $\{5, 10, 15, \ldots, 95, 100\}$
(a) $Pr(A \cup B) = Pr(A) + Pr(B) - Pr(A \cap B) = \frac{50}{100} + \frac{33}{100} - \frac{16}{100} = \frac{67}{100} = 0.67$. [Note: Here $A \cap B = \{6, 12, 18, \ldots, 96\}$, the set of integers between 1 and 100 (inclusive) that are divisible by 6 – that is, divisible by both 2 and 3.]
(b) $Pr(A \cup B \cup C) = Pr(A) + Pr(B) + Pr(C) - Pr(A \cap B) - Pr(A \cap C) - Pr(B \cap C) + Pr(A \cap B \cap C) = \frac{50}{100} + \frac{33}{100} + \frac{20}{100} - \frac{16}{100} - \frac{10}{100} - \frac{6}{100} + \frac{3}{100} = \frac{74}{100} = 0.74$.

37. A common divisor for m, n has the form $p_1^{r_1} p_2^{r_2} p_3^{r_3}$, where $0 \leq r_i \leq \min\{e_i, f_i\}$, for all $1 \leq i \leq 3$. Let $m_i = \min\{e_i, f_i\}$, $1 \leq i \leq 3$. Then the number of common divisors is $(m_1 + 1)(m_2 + 1)(m_3 + 1)$.

CHAPTER 5
RELATIONS AND FUNCTIONS

Section 5.1

1. $A \times B = \{(1,2),(2,2),(3,2),(4,2),(1,5),(2,5),(3,5),(4,5)\}$
$B \times A = \{(2,1),(2,2),(2,3),(2,4),(5,1),(5,2),(5,3),(5,4)\}$
$A \cup (B \times C) = \{1,2,3,4,(2,3),(2,4),(2,7),(5,3),(5,4),(5,7)\}$
$(A \cup B) \times C = (A \times C) \cup (B \times C) = \{(1,3),(2,3),(3,3),(4,3),(5,3),(1,4),(2,4),(3,4),(4,4),(5,4),(1,7),(2,7),(3,7),(4,7),(5,7)\}$

2. (a) $\{(1,2)\}; \{(1,2),(1,4),(1,5),(2,2),(2,4)\}; A \times B$
(b) $\{(1,1),(2,2),(3,3)\}; \{(1,1),(1,2),(1,3),(2,2),(3,3)\}; \{(1,2),(2,1),(3,3)\}$.

3. (a) $|A \times B| = |A||B| = 9$
(b) Since a relation from A to B is a subset of $A \times B$, there are 2^9 relations from A to B.
(c) Since $|A \times A| = 9$, there are 2^9 relations on A.
(d) For the other seven ordered pairs in $A \times B$ there are two choices: include it in the relation or leave it out. Hence there are 2^7 relations from A to B that contain (1,2) and (1,5).
(e) $\binom{9}{5}$ (f) $\binom{9}{7} + \binom{9}{8} + \binom{9}{9}$

4. If either A or B is $\emptyset$ and when $A = B$.

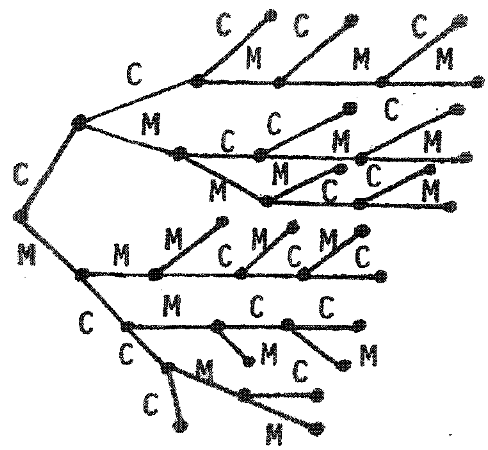
5. (a) Assume that $A \times B \subseteq C \times D$ and let $a \in A$ and $b \in B$. Then $(a,b) \in A \times B$, and since $A \times B \subseteq C \times D$ we have $(a,b) \in C \times D$. But $(a,b) \in C \times D \Rightarrow a \in C$ and $b \in D$. Hence, $a \in A \Rightarrow a \in C$, so $A \subseteq C$, and $b \in B \Rightarrow b \in D$, so $B \subseteq D$.
Conversely, suppose that $A \subseteq C$ and $B \subseteq D$, and that $(x,y) \in A \times B$. Then $(x,y) \in A \times B \Rightarrow x \in A$ and $y \in B \Rightarrow x \in C$ (since $A \subseteq C$) and $y \in D$ (since $B \subseteq D$) $\Rightarrow (x,y) \in C \times D$. Consequently, $A \times B \subseteq C \times D$.

(b) If any of the sets A, B, C, D is empty we still find that

$$[(A \subseteq C) \wedge (B \subseteq D)] \Rightarrow [A \times B \subseteq C \times D].$$

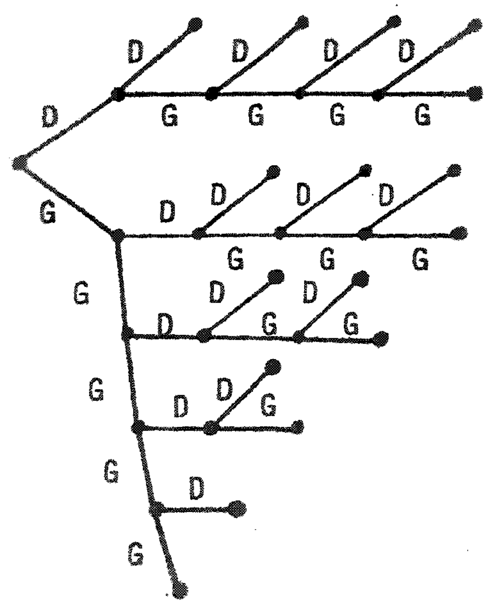
However, the converse need not hold. For example, let $A = \emptyset$, $B = \{1,2\}$, $C = \{1,2\}$ and $D = \{1\}$. Then $A \times B = \emptyset$ — if not, there exists an ordered pair (x,y) in $A \times B$, and this means that the empty set A contains an element x. And so $A \times B = \emptyset \subseteq C \times D$ — but $B = \{1,2\} \not\subseteq \{1\} = D$.

6.



7. (a) Since $|A| = 5$ and $|B| = 4$ we have $|A \times B| = |A||B| = 5 \cdot 4 = 20$. Consequently, $A \times B$ has 2^{20} subsets, so $|\mathcal{P}(A \times B)| = 2^{20}$.
(b) If $|A| = m$ and $|B| = n$, for $m, n \in \mathbf{N}$, then $|A \times B| = mn$. Consequently, $|\mathcal{P}(A \times B)| = 2^{mn}$.

8.



9. (b) $A \times (B \cup C) = \{(x,y)|x \in A \text{ and } y \in (B \cup C)\} = \{(x,y)|x \in A \text{ and } (y \in B \text{ or } y \in C)\} = \{(x,y)|(x \in A \text{ and } y \in B) \text{ or } (x \in A \text{ and } y \in C)\} = \{(x,y)|x \in A \text{ and } y \in B\} \cup \{(x,y)|x \in A \text{ and } y \in C\} = (A \times B) \cup (A \times C)$.
(c) & (d) The proofs here are similar to that given in part (b).

10. $1 + 2 + 2(3) + 2(3)(5) = 39;\ 38$

11. $(x,y) \in A \times (B - C) \iff x \in A \text{ and } y \in B - C \iff x \in A \text{ and } (y \in B \text{ and } y \notin C) \iff (x \in A \text{ and } y \in B) \text{ and } (x \in A \text{ and } y \notin C) \iff (x,y) \in A \times B \text{ and } (x,y) \notin A \times C \iff (x,y) \in (A \times B) - (A \times C)$..

12. $2^{(3|B|)} = 4096 \Longrightarrow 3|B| = 12 \Longrightarrow |B| = 4$.

13. (a) (1) $(0,2) \in \mathcal{R}$; and
(2) If $(a,b) \in \mathcal{R}$, then $(a+1, b+5) \in \mathcal{R}$.
(b) From part (1) of the definition we have $(0,2) \in \mathcal{R}$. By part (2) of the definition we then find that
(i) $(0,2) \in \mathcal{R} \Rightarrow (0+1, 2+5) = (1,7) \in \mathcal{R}$;
(ii) $(1,7) \in \mathcal{R} \Rightarrow (1+1, 7+5) = (2,12) \in \mathcal{R}$;
(iii) $(2,12) \in \mathcal{R} \Rightarrow (2+1, 12+5) = (3,17) \in \mathcal{R}$; and
(iv) $(3,17) \in \mathcal{R} \Rightarrow (3+1, 17+5) = (4,22) \in \mathcal{R}$.

14. (a) (1) $(1,1), (2,1) \in \mathcal{R}$; and
(2) If $(a,b) \in \mathcal{R}$, then $(a+1, b+1)$ and $(a+1, b)$ are in $\mathcal{R}$.
(b) Start with (2,1) in $\mathcal{R}$ — from part (1) of the definition. Then by part (2) we get
(i) $(2,1) \in \mathcal{R} \Rightarrow (2+1, 1+1) = (3,2) \in \mathcal{R}$;
(ii) $(3,2) \in \mathcal{R} \Rightarrow (3+1, 2) = (4,2) \in \mathcal{R}$; and
(iii) $(4,2) \in \mathcal{R} \Rightarrow (4+1, 2) = (5,2) \in \mathcal{R}$.

Start with (1,1) in $\mathcal{R}$ — from part (1) of the definition. Then we find from part (2) that
(i) $(1,1) \in \mathcal{R} \Rightarrow (1+1, 1+1) = (2,2) \in \mathcal{R}$;
(ii) $(2,2) \in \mathcal{R} \Rightarrow (2+1, 2+1) = (3,3) \in \mathcal{R}$; and
(iii) $(3,3) \in \mathcal{R} \Rightarrow (3+1, 3+1) = (4,4) \in \mathcal{R}$.

Section 5.2

1. (a) Function: Range $= \{7, 8, 11, 16, 23, \ldots\}$
(b) Relation, not a function. For example, both (4,2) and $(4, -2)$ are in the relation.
(c) Function: Range = the set of all real numbers.
(d) Relation, not a function. Both (0,1) and $(0, -1)$ are in the relation.
(e) Since $|R| > 5$, R cannot be a function.

2. The formula cannot be used for the domain of real numbers since $f(\sqrt{2})$, $f(-\sqrt{2})$ are undefined. Since $\sqrt{2}, -\sqrt{2} \notin \mathbf{Z}$ the formula does define a real valued function on the domain $\mathbf{Z}$.

3. (a) $\{(1,x),(2,x),(3,x),(4,x)\}, \{(1,y),(2,y),(3,y),(4,y)\}, \{(1,z),(2,z),(3,z),(4,z)\}$
$\{(1,x),(2,y),(3,x),(4,y)\}, \{(1,x),(2,y),(3,z),(4,x)\}$
(b) 3^4 (c) 0 (d) 4^3 (e) 24 (f) 3^3 (g) 3^2 (h) 3^2

4. $3^{|A|} = 2187 \Longrightarrow |A| = 7$

5. (a) $A \cap B = \{(x,y)|y = 2x+1 \text{ and } y = 3x\}$
$2x + 1 = 3x \Rightarrow x = 1$
So $A \cap B = \{(1,3)\}$.
(b) $B \cap C = \{(x,y)|y = 3x \text{ and } y = x - 7\}$
$3x = x - 7 \Rightarrow 2x = -7$, so $x = -7/2$.
Consequently, $B \cap C = \{(-7/2, 3(-7/2)\} = \{(-7/2, -21/2)\}$.
(c) $\overline{\overline{A} \cup \overline{C}} = \overline{\overline{A}} \cap \overline{\overline{C}} = A \cap C = \{(x,y)|y = 2x+1 \text{ and } y = x - 7\}$
Now $2x + 1 = x - 7 \Rightarrow x = -8$, and so $A \cap C = \{(-8, -15)\}$.
(d) We know that $\overline{B} \cup \overline{C} = \overline{B \cap C}$, and since $B \cap C = \{(-7/2, -21/2)\}$ we have $\overline{B} \cup \overline{C} = \mathbf{R}^2 - \{(-7/2, -21/2)\} = \{(x,y)|x \neq -7/2 \text{ or } y \neq -21/2\}$.

6.
(a) (i) $A \cap B = \{(1,3)\}$ (ii) $B \cap C = \{\} = \emptyset$
(iii) $\overline{\overline{A} \cup \overline{C}} = \{(-8,-15)\}$ (iv) $\overline{B} \cup \overline{C} = \mathbf{Z}^2 = \mathbf{Z} \times \mathbf{Z}$

(b) (i) $A \cap B = \{(1,3)\}$ (ii) $B \cap C = \{\} = \emptyset$
(iii) $\overline{\overline{A} \cup \overline{C}} = \emptyset$ (iv) $\overline{B} \cup \overline{C} = \mathbf{Z}^+ \times \mathbf{Z}^+$

7.
(a) $\lfloor 2.3 - 1.6 \rfloor = \lfloor 0.7 \rfloor = 0$ (b) $\lfloor 2.3 \rfloor - \lfloor 1.6 \rfloor = 2 - 1 = 1$
(c) $\lceil 3.4 \rceil \lfloor 6.2 \rfloor = 4 \cdot 6 = 24$ (d) $\lfloor 3.4 \rfloor \lceil 6.2 \rceil = 3 \cdot 7 = 21$
(e) $\lfloor 2\pi \rfloor = 6$ (f) $2\lceil \pi \rceil = 8$

8. (a) True (b) False: Let $a = 1.5$. Then $\lfloor 1.5 \rfloor = 1 \neq 2 = \lceil 1.5 \rceil$
(c) True (d) False: Let $a = 1.5$. Then $-\lceil a \rceil = -2 \neq -1 = \lceil -a \rceil$.

9. (a) $\ldots [-1, -6/7) \cup [0, 1/7) \cup [1, 8/7) \cup [2, 15/7) \cup \ldots$
(b) $[1, 8/7)$ (c) $\mathbf{Z}$ (d) $\mathbf{R}$

10. $\mathbf{R}$

11. (a) $\ldots \cup (-7/3, -2] \cup (-4/3, -1] \cup (-1/3, 0] \cup (2/3, 1] \cup (5/3, 2] \cup \ldots = \bigcup_{m \in \mathbf{Z}^+} (m - 1/3, m]$

(b) $\ldots \cup ((-2n-1)/n, -2] \cup ((-n-1)/n, -1] \cup (-1/n, 0] \cup ((n-1)/n, 1] \cup ((2n-1)/n, 2] \cup \ldots$

$= \bigcup_{m \in \mathbf{Z}^+} (m - 1/n, m]$

12. Proof: (Case 1: $k|n$) Here $n = qk$ for $q \in \mathbf{Z}^+$, and $(n-1)/k = (qk-1)/k = q - (1/k)$ with $q - 1 \leq q - (1/k) < q$. Therefore $\lceil n/k \rceil = \lceil q \rceil = q = (q-1)+1 = \lfloor (n-1)/k \rfloor + 1$.
(Case 2: $k \not| n$) Now we have $n = qk + r$, where $q, r \in \mathbf{Z}^+$ with $r < k$, and $n/k = q + (r/k)$ with $0 < (r/k) < 1$. So $n - 1 = qk + (r-1)$ and $(n-1)/k = q + [(r-1)/k]$ with $0 \leq [(r-1)/k] < 1$. Consequently, $\lceil n/k \rceil = \lceil q + (r/k) \rceil = q + 1 = \lfloor (n-1)/k \rfloor + 1$.

13. a) Proof (i): If $a \in \mathbf{Z}^+$, then $\lceil a \rceil = a$ and $\lfloor \lceil a \rceil / a \rfloor = \lfloor 1 \rfloor = 1$. If $a \notin \mathbf{Z}^+$, write $a = n + c$, where $n \in \mathbf{Z}^+$ and $0 < c < 1$. Then $\lceil a \rceil / a = (n+1)/(n+c) = 1 + (1-c)/(n+c)$, where $0 < (1-c)/(n+c) < 1$. Hence $\lfloor \lceil a \rceil / a \rfloor = \lfloor 1 + (1-c)/(n+c) \rfloor = 1$.

Proof (ii): For $a \in \mathbf{Z}^+$, $\lfloor a \rfloor = a$ and $\lceil \lfloor a \rfloor / a \rceil = \lceil 1 \rceil = 1$. When $a \notin \mathbf{Z}^+$, let $a = n + c$, where $n \in \mathbf{Z}^+$ and $0 < c < 1$. Then $\lfloor a \rfloor / a = n/(n+c) = 1 - [c/(n+c)]$, where $0 < c/(n+c) < 1$. Consequently $\lceil \lfloor a \rfloor / a \rceil = \lceil 1 - (c/(n+c)) \rceil = 1$.

b) Consider $a = 0.1$. Then
(i) $\lfloor \lceil a \rceil / a \rfloor = \lfloor 1/0.1 \rfloor = \lfloor 10 \rfloor = 10 \neq 1$; and
(ii) $\lceil \lfloor a \rfloor / a \rceil = \lceil 0/0.1 \rceil = 0 \neq 1$.

In fact (ii) is false for all $0 < a < 1$, since $\lceil \lfloor a \rfloor / a \rceil = 0$ for all such values of a. In the case of (i), when $0 < a \leq 0.5$, it follows that $\lceil a \rceil / a \geq 2$ and $\lfloor \lceil a \rceil / a \rfloor \geq 2 \neq 1$. However, for $0.5 < a < 1$, $\lceil a \rceil / a = 1/a$ where $1 < 1/a < 2$, and so $\lfloor \lceil a \rceil / a \rfloor = 1$ for $0.5 < a < 1$.

14.

(a)
$$\begin{aligned} a_2 &= 2a_{\lfloor 2/2 \rfloor} = 2a_1 = 2 \\ a_3 &= 2a_{\lfloor 3/2 \rfloor} = 2a_1 = 2 \\ a_4 &= 2a_{\lfloor 4/2 \rfloor} = 2a_2 = 4 \\ a_5 &= 2a_{\lfloor 5/2 \rfloor} = 2a_2 = 4 \\ a_6 &= 2a_{\lfloor 6/2 \rfloor} = 2a_3 = 4 \\ a_7 &= 2a_{\lfloor 7/2 \rfloor} = 2a_3 = 4 \\ a_8 &= 2a_{\lfloor 8/2 \rfloor} = 2a_4 = 8 \end{aligned}$$

(b) Proof: (By the Alternative Form of the Principle of Mathematical Induction)
For $n = 1$ we have $a_1 = 1 \leq 1$, so the result is true in this first case. (This provides the basis step for the proof.)
Now assume the result true for some $k \geq 1$ and all $n = 1, 2, 3, \ldots, k-1, k$. For $n = k+1$ we have $a_{k+1} = 2a_{\lfloor (k+1)/2 \rfloor} \leq 2\lfloor (k+1)/2 \rfloor$, where the inequality follows from the assumption of the induction hypothesis.
When k is odd, then, $\lfloor (k+1)/2 \rfloor = (k+1)/2$ and we have $a_{k+1} \leq 2[(k+1)/2] = k+1$.
When k is even, then $\lfloor (k+1)/2 \rfloor = \lfloor (k/2) + (1/2) \rfloor = (k/2)$, and here we find that $a_{k+1} \leq 2(k/2) = k \leq k+1$.
In either case it follows from $a_{\lfloor (k+1)/2 \rfloor} \leq \lfloor (k+1)/2 \rfloor$ that $a_{k+1} \leq k+1$. So we have established the inductive step of the proof.
Therefore, it follows from the Alternative Form of the Principle of Mathematical Induction that

$$\forall n \in \mathbf{Z}^+ \quad a_n \leq n.$$

15. (a) One-to-one. The range is the set of all odd integers.
(b) One-to-one. Range = $\mathbf{Q}$
(c) Since $f(1) = f(0), f$ is not one-to-one. The range of $f = \{0, \pm 6, \pm 24, \pm 60, \ldots\} = \{n^3 - n | n \in \mathbf{Z}\}$.
(d) One-to-one. Range = $(0, +\infty) = \mathbf{R}^+$
(e) One-to-one. Range = $[-1, 1]$
(f) Since $f(\pi/4) = f(3\pi/4), f$ is not one-to-one. The range of $f = [0, 1]$.

16.

(a) {4,9}	(b) {4,9}	(c) [0,9)
(d) [0,9)	(e) [0,49]	(f) $[9, 16) \cup [25, 36]$

17. The extension must include $f(1)$ and $f(4)$. Since $|B| = 4$ there are four choices for each of 1 and 4, so there are $4^2 = 16$ ways to extend the given function g.

18. Let $A = \{1,2\}, B = \{3,4\}$ and $f = \{(1,3),(2,3)\}$. For $A_1 = \{1\}, A_2 = \{2\}$, $f(A_1 \cap A_2) = f(\emptyset) = \emptyset$ while $f(A_1) \cap f(A_2) = \{3\} \cap \{3\} = \{3\}$.

19. (a) $f(A_1 \cup A_2) = \{y \in B | y = f(x),\ x \in A_1 \cup A_2\} = \{y \in B | y = f(x),\ x \in A_1$ or $x \in A_2\} = \{y \in B | y = f(x),\ x \in A_1\} \cup \{y \in B | y = f(x),\ x \in A_2\} = f(A_1) \cup f(A_2)$.
(c) $y \in f(A_1) \cap f(A_2) \Longrightarrow y = f(x_1) = f(x_2),\ x_1 \in A_1,\ x_2 \in A_2 \Longrightarrow y = f(x_1)$ with $x_1 = x_2$, since f is one-to-one $\Longrightarrow y \in f(A_1 \cap A_2)$.

20. The number of injective (or, one-to-one) functions from A to B is $(|B|!)/(|B| - 5)! = 6720$, and $|B| = 8$.

21. No. Let $A = \{1,2\}, X = \{1\}, Y = \{2\}, B = \{3\}$. For $f = \{(1,3),(2,3)\}$ we have $f|_X$, $f|_Y$ one-to-one, but f is not one-to-one.

22. (a) A monotone increasing function $f : X_7 \to X_5$ determines a selection, with repetitions allowed, of size 7 from {1,2,3,4,5}, and vice versa. For example, the selection 1,1,2,2,3,5,5 corresponds to the monotone increasing function $g : X_7 \to X_5$, where $g = \{(1,1),(2,1),(3,2),(4,2),(5,3),(6,5),(7,5)\}$. (Note the second components.) Consequently, the number of monotone increasing functions $f : X_7 \to X_5$ is $\binom{5+7-1}{7} = \binom{11}{7} = 330$.

(b) $\binom{9+6-1}{6} = \binom{14}{6} = 3003$.

(c) For $m, n \in \mathbf{Z}^+$, the number of monotone increasing functions $f : X_m \to X_n$ is $\binom{n+m-1}{m}$.

(d) Since $f(4) = 4$, it follows that $f(\{1,2,3\}) \subseteq \{1,2,3,4\}$ and $f(\{5,6,7,8,9,10\}) \subseteq \{4,5,6,7,8\}$ because f is monotone increasing. The number of these functions is $\binom{4+3-1}{3}\binom{5+6-1}{6} = \binom{6}{3}\binom{10}{6} = (20)(210) = 4200$.

(e) $\binom{12}{4}\binom{5}{2} = 4950$.

(f) Let $m, n, k, \ell \in \mathbf{Z}^+$ with $1 \leq k \leq m$ and $1 \leq \ell \leq n$. If $f : X_m \to X_n$ is monotone increasing and $f(k)=\ell$, then $f(\{1,2,\ldots,k-1\}) \subseteq \{1,2,\ldots,\ell\}$ and $f(\{k+1,\ldots,m\}) \subseteq \{\ell,\ell+1,\ldots,n\}$. So there are $\binom{\ell+(k-1)-1}{(k-1)}\binom{(n-\ell+1)+(m-(k+1)+1)-1}{(m-(k+1)+1)} = \binom{\ell+k-2}{k-1}\binom{n+m-\ell-k}{m-k}$ such functions.

23. (a) $f(a_{ij}) = 12(i-1)+j$ (b) $f(a_{ij}) = 10(i-1)+j$ (c) $f(a_{ij}) = 7(i-1)+j$

24. $g(a_{ij}) = m(j-1)+i$

25. (a) (i) $f(a_{ij}) = n(i-1)+(k-1)+j$
(ii) $g(a_{ij}) = m(j-1)+(k-1)+i$
(b) $k+(mn-1) \leq r$

26. (a) There is only one function in S_1, namely $f: A \to B$ where $f(a) = f(b) = 1$ and $f(c)=2$. Hence $|S_1| = 1$.
(b) Since $f(c)=3$ we have two choices — namely 1,2 — for each of $f(a)$ and $f(b)$. Consequently, $|S_2| = 2^2$.
(c) With $f(c) = i+1$ there are i choices — namely $1,2,3,\ldots,i-1,i$ — for each of $f(a)$ and $f(b)$, so $|S_i| = i^2$.
(d) Any function f in T_1 is determined by two elements x, y in B, where $1 \leq x < y \leq n+1$ and $f(a)=f(b)=x$, $f(c)=y$. We can select these two elements from B in $\binom{n+1}{2}$ ways, so $|T_1| = \binom{n+1}{2}$.
(e) For T_2 we have $f(a) < f(b) < f(c)$, so we need three distinct elements from B, and these can be chosen in $\binom{n+1}{3}$ ways. The argument for T_3 is similar.
(f) $S = S_1 \cup S_2 \cup S_3 \cup \ldots \cup S_n$, where $S_i \cap S_j = \emptyset$ for all $1 \leq i < j \leq n$, and $S = T_1 \cup T_2 \cup T_3$ with $T_1 \cap T_2 = T_1 \cap T_3 = T_2 \cap T_3 = \emptyset$.
(g) From part (f) we have $|S| = \sum_{i=1}^{n}|S_i| = \sum_{i=1}^{n} i^2 = \sum_{j=1}^{3}|T_j| = \binom{n+1}{2} + 2\binom{n+1}{3}$. Hence

$\sum_{i=1}^{n} i^2 = (n+1)(n)/2 \ + \ 2(n+1)(n)(n-1)/6 = (n+1)(n)[(1/2)+(n-1)/3] =$
$(n+1)(n)[(3+2n-2)/6] = n(n+1)(2n+1)/6$.

27. (a) $A(1,3) = A(0,A(1,2)) = A(1,2)+1 = A(0,A(1,1))+1 = [A(1,1)+1]+1 = A(1,1)+2 = A(0,A(1,0))+2 = [A(1,0)+1]+2 = A(1,0)+3 = A(0,1)+3 = (1+1)+3 = 5$

$A(2,3) = A(1,A(2,2))$
$A(2,2) = A(1,A(2,1))$
$A(2,1) = A(1,A(2,0)) = A(1,A(1,1))$
$A(1,1) = A(0,A(1,0)) = A(1,0)+1 = A(0,1)+1 = (1+1)+1 = 3$
$A(2,1) = A(1,3) = A(0,A(1,2)) = A(1,2)+1 = A(0,A(1,1)) = [A(1,1)+1]+1 = 5$
$A(2,2) = A(1,5) = A(0,A(1,4)) = A(1,4)+1 = A(0,A(1,3))+1 = A(1,3)+2 = A(0,A(1,2))+2 = A(1,2)+3 = A(0,A(1,1))+3 = A(1,1)+4 = 7$
$A(2,3) = A(1,7) = A(0,A(1,6)) = A(1,6)+1 = A(0,A(1,5))+1 = A(0,7)+1 = (7+1)+1 = 9$

(b) Since $A(1,0) = A(0,1) = 2 = 0+2$, the result holds for the case where $n = 0$. Assuming the truth of the (open) statement for some k (≥ 0) we have $A(1,k) = k+2$. Then we find that $A(1,k+1) = A(0, A(1,k)) = A(1,k)+1 = (k+2)+1 = (k+1)+2$, so the truth at $n = k$ implies the truth at $n = k+1$. Consequently, $A(1,n) = n+2$ for all $n \in \mathbf{N}$ by the Principle of Mathematical Induction.

(c) Here we find that $A(2,0) = A(1,1) = 1+2 = 3$ (by the result in part(b)). So $A(2,0) = 3 + 2 \cdot 0$ and the given (open) statement is true in this first case.
Next we assume the result true for some k (≥ 0) — that is, we assume that $A(2,k) = 3+2k$. For $k+1$ we then find that $A(2,k+1) = A(1, A(2,k)) = A(2,k)+2$ (by part (b)) $=$ $(3+2k)+2$ (by the induction hypothesis) $= 3+2(k+1)$. Consequently, for all $n \in \mathbf{N}$, $A(2,n) = 3+2n$ — by the Principle of Mathematical Induction.

(d) Once again we consider what happens for $n = 0$. Since $A(3,0) = A(2,1) = 3+2(1)$ (by part (c)) $= 5 = 2^{0+3} - 3$, the result holds in this first case.
So now we assume the given (open) statement is true for some k (≥ 0) and this gives us the induction hypothesis: $A(3,k) = 2^{k+3} - 3$. For $n = k+1$ it then follows that $A(3,k+1) = A(2, A(3,k)) = 3 + 2A(3,k)$ (by part (c)) $= 3 + 2(2^{k+3} - 3)$ (by the induction hypothesis) $= 2^{(k+1)+3} - 3$, so the result holds for $n = k+1$ whenever it does for $n = k$. Therefore, $A(3,n) = 2^{n+3} - 3$, for all $n \in \mathbf{N}$ — by the Principle of Mathematical Induction.

28. (a) $\binom{5}{4}4^4 + \binom{5}{3}4^3 + \binom{5}{2}4^2 + \binom{5}{1}4^1 = (4+1)^5 - \binom{5}{5}4^5 - \binom{5}{0}4^0 = 5^5 - 4^5 - 1$

(b) $\binom{m}{m-1}n^{m-1} + \binom{m}{m-2}n^{m-2} + \cdots + \binom{m}{1}n^1 = (n+1)^m - n^m - 1$.

Section 5.3

1. Let $A = \{1,2,3,4\}, B = \{v,w,x,y,z\}$: (a) $f = \{(1,v),(2,v),(3,w),(4,x)\}$
(b) $f = \{(1,v),(2,x),(3,y),(4,z)\}$
(c) Let $A = \{1,2,3,4,5\}, B = \{w,x,y,z\}, f = \{(1,w),(2,w),(3,x),(4,y),(5,z)\}$.
(d) Let $A = \{1,2,3,4\}, B = \{w,x,y,z\}, f = \{(1,w),(2,x),(3,y),(4,z)\}$.

2. (a) One-to-one and onto.
(b) One-to-one but not onto. The range consists of all the odd integers.
(c) One-to-one and onto.
(d) Since $f(-1) = f(1)$, f is not one-to-one. Also f is not onto. The range of $f = \{0,1,4,9,16,\ldots\}$.
(e) Since $f(0) = f(-1)$, f is not one-to-one. Also f is not onto. The range of $f = \{0,2,6,12,20,\ldots\}$.
(f) One-to-one but not onto. The range of $f = \{\ldots,-64,-27,-8,-1,0,1,8,27,\ldots\}$.

3. (a), (b), (c), (f) One-to-one and onto.
(d) Neither one-to-one nor onto. Range $= [0,+\infty)$
(e) Neither one-to-one nor onto. Range $= [-1/4,+\infty)$

4. (a) 6^4; $6!/2!$; 0 (b) 4^6; $(4!)S(6,4)$; 0

5. For $n=5, m=3, \sum_{k=0}^{5}(-1)^k\binom{5}{5-k}(5-k)^3 = (-1)^0\binom{5}{5}(5)^3+(-1)^1\binom{5}{4}(4)^3+(-1)^2\binom{5}{3}(3)^3+(-1)^3\binom{5}{2}(2)^3+(-1)^4\binom{5}{1}(1)^3+(-1)^5\binom{5}{0}(0)^3 = 125-320+70-80+5=0$

6. (a) $\sum_{i=1}^{5}\binom{5}{i}(i!)S(7,i) = \binom{5}{1}(1!)S(7,1)+\binom{5}{2}(2!)S(7,2)+\binom{5}{3}(3!)S(7,3)+\binom{5}{4}(4!)S(7,4)+\binom{5}{5}(5!)S(7,5) = (5)(1)(1)+(10)(2)(63)+(10)(6)(301)+(5)(24)(350)+(1)(120)(14) = 78,125 = 5^7$.

(b) The expression m^n counts the number of ways to distribute n distinct objects among m distinct containers.

For $1 \le i \le m$, let i count the number of distinct containers that we actually use — that is, those that are *not* empty after the n distinct objects are distributed. This number of distinct containers can be chosen in $\binom{m}{i}$ ways. Once we have the i distinct containers we can distribute the n distinct objects among these i distinct containers, with no container left empty, in $(i!)S(n,i)$ ways — where $S(n,i)=0$ when $n<i$. Then $\sum_{i=1}^{m}\binom{m}{i}(i!)S(n,i)$ also counts the number of ways to distribute n distinct objects among m distinct containers.

Hence $m^n = \sum_{i=1}^{m}\binom{m}{i}(i!)S(n,i)$.

7. (a) (i) $2!S(7,2)$ (ii) $\binom{5}{2}[2!S(7,2)]$ (iii) $3!S(7,3)$
(iv) $\binom{5}{3}[3!S(7,3)]$ (v) $4!S(7,4)$ (vi) $\binom{5}{4}[4!S(7,4)]$

(b) $\binom{n}{k}[k!S(m,k)]$

8. Let A be the set of compounds and B the set of assistants. Then the number of assignments with no idle assistants is the number of onto functions from set A to set B. There are $5!S(9,5)$ such functions.

9. For each $r \in \mathbf{R}$ there is at least one $a \in \mathbf{R}$ such that $a^5-2a^2+a-r=0$ because the polynomial x^5-2x^2+x-r has odd degree and real coefficients. Consequently, f is onto. However, $f(0)=0=f(1)$, so f is not one-to-one.

10. (a) $(4!)S(7,4)$
(b) $(3!)S(6,3)$ (Here container II contains only the blue ball) $+$ $(4!)S(6,4)$ (Here container II contains more than just the blue ball).
(c) $S(7,4)+S(7,3)+S(7,2)+S(7,1)$.

11.

m \ n	1	2	3	4	5	6	7	8	9	10
9	1	255	3025	7770	6951	2646	462	36	1	
10	1	511	9330	34105	42525	22827	5880	750	45	1

12. (a) Since $31,100,905 = 5 \times 11 \times 17 \times 29 \times 31 \times 37$, we find that there are $S(6,3) = 90$ unordered factorizations of 31,100,905 into three factors — each greater than 1.
(b) If the order of the factors in part (a) is considered relevant then there are $(3!)S(6,3) = 540$ such factorizations.
(c) $\sum_{i=2}^{6} S(6,i) = S(6,2) + S(6,3) + S(6,4) + S(6,5) + S(6,6) = 31 + 90 + 65 + 15 + 1 = 202$
(d) $\sum_{i=2}^{6} (i!)S(6,i) = (2!)S(6,2) + (3!)S(6,3) + (4!)S(6,4) + (5!)S(6,5) + (6!)S(6,6) =$

$$(2)(31) + (6)(90) + (24)(65) + (120)(15) + (720)(1) = 4682.$$

13. (a) Since $156,009 = 3 \times 7 \times 17 \times 19 \times 23$, it follows that there are $S(5,2) = 15$ two-factor unordered factorizations of 156,009, where each factor is greater than 1.
(b) $\sum_{i=2}^{5} S(5,i) = S(5,2) + S(5,3) + S(5,4) + S(5,5) = 15 + 25 + 10 + 1 = 51.$
(c) $\sum_{i=2}^{n} S(n,i).$

14.

```
10    Dim S(12, 12)
20    For I = 1 To 12
30          S(I,I) = 1
40    Next I
50    Print "M =  :   1"
60    For M = 2 To 12
70          Print "M ="; M; ":  1, ";
80          For N = 2 To M-1
90                S(M,N) = S(M-1,N-1) + N*S(M-1,N)
100               Print S(M,N); ", ";
110         Next N
120         Print "  1"
130   Next M
140   End
```

15. a) $n = 4$: $\sum_{i=1}^{4} i!S(4,i)$; $n = 5$: $\sum_{i=1}^{5} i!S(5,i)$

In general, the answer is $\sum_{i=1}^{n} i!S(n,i)$.

b) $\binom{15}{12}\sum_{i=1}^{12} i!S(12,i)$.

16. a) (i) 10!

(ii) The given outcome — namely, $\{C_2, C_3, C_7\}$, $\{C_1, C_4, C_9, C_{10}\}$, $\{C_5\}$, $\{C_6, C_8\}$ — is an example of a distribution of ten distinct objects among four distinct containers, with no container left empty. [Or it is an example of an onto function $f : A \to B$ where $A = \{C_1, C_2, \ldots, C_{10}\}$ and $B = \{1,2,3,4\}$.] There are $4!S(10,4)$ such distributions [or functions].

The answer to the question is $\sum_{i=1}^{10} i!S(10,i)$.

(iii) $\binom{10}{3}\sum_{i=1}^{7} i!S(7,i)$.

b) $\binom{9}{2}\sum_{i=1}^{7} i!S(7,i)$

c) For $0 \le k \le 9$, the number of outcomes where C_3 is tied for first place with k other candidates is $\binom{9}{k}\sum_{i=1}^{9-k} i!S(9-k,i)$. [Part (b) above is the special case where $k = 3 - 1 = 2$.]

Summing over the possible values of k we have the answer $\sum_{k=0}^{9}\binom{9}{k}\sum_{i=1}^{9-k} i!S(9-k,i)$.

17. Let $a_1, a_2, \ldots, a_m, x$ denote the $m+1$ distinct objects. Then $S_r(m+1,n)$ counts the number of ways these objects can be distributed among n identical containers so that each container receives at least r of the objects.

Each of these distributions falls into exactly one of two categories:

1) The element x is in a container with r or more other objects: Here we start with $S_r(m,n)$ distributions of $a_1, a_2, \ldots, a_m$ into n identical containers – each container receiving at least r of the objects. Now we have n ***distinct*** containers – distinguished by their contents. Consequently, there are n choices for locating the object x. As a result, this category provides $nS_r(m,n)$ of the distributions.

2) The element x is in a container with $r-1$ of the other objects: These other $r-1$ objects can be chosen in $\binom{m}{r-1}$ ways, and then these objects – along with x – can be placed in one of the n containers. The remaining $m+1-r$ distinct objects can then be distributed among the $n-1$ identical containers – where each container receives at least r of the objects – in $S_r(m+1-r, n-1)$ ways. Hence this category provides the remaining $\binom{m}{r-1}S_r(m+1-r, n-1)$ distributions.

18. (a) For $n > m$ we have $s(m,n) = 0$, because there are more tables than people.

(b) For $m \geq 1$, (i) $s(m,m) = 1$ because the ordering of the m tables is not taken into account; and, (ii) $s(m,1) = (m-1)!$, as in Example 1.16.
(c) Here there are two people at one table and one at each of the other $m-1$ tables. There are $\binom{m}{2}$ such arrangements.
(d) When m people are seated around $m-2$ tables there are two cases to consider: (1) One table with three occupants and $m-3$ tables, each with one occupant — there are $\binom{m}{3}(2!)$ such arrangements; and, (2) Two tables, each with two occupants, and $m-4$ tables each with a single occupant — there are $(1/2)\binom{m}{2}\binom{m-2}{2}$ of these arrangements. We then find that $\binom{m}{3}(2!)+(1/2)\binom{m}{2}\binom{m-2}{2} = (1/3)(m)(m-1)(m-2)+(1/2)[(1/2)(m)(m-1)][(1/2)(m-2)(m-3)] = (m)(m-1)(m-2)[(1/3)+(1/8)(m-3)] = (1/24)(m)(m-1)(m-2)(3m-1)$.

19. (a) We know that $s(m,n)$ counts the number of ways we can place m people — call them $p_1, p_2, \ldots, p_m$ — around n circular tables, with at least one occupant at each table. These arrangements fall into two disjoint sets: (1) The arrangements where p_1 is alone: There are $s(m-1, n-1)$ such arrangements; and, (2) The arrangements where p_1 shares a table with at least one of the other $m-1$ people: There are $s(m-1,n)$ ways where $p_2, p_3, \ldots, p_m$ can be seated around the n tables so that every table is occupied. Each such arrangement determines a total of $m-1$ locations (at all the n tables) where p_1 can now be seated — this for a total of $(m-1)s(m-1,n)$ arrangements. Consequently, $s(m,n) = (m-1)s(m-1,n) + s(m-1,n-1)$, for $m \geq n > 1$.

(b) For $m = 2$, we have $s(m,2) = 1 = 1!(1/1) = (m-1)!\sum_{i=1}^{m-1} \frac{1}{i}$. So the result is true in this case; this establishes the basis step for a proof by mathematical induction. Assuming the result for $m = k(\geq 2)$ we have $s(k,2) = (k-1)!\sum_{i=1}^{k-1} \frac{1}{i}$. Using the result from part (a) we now find that $s(k+1,2) = ks(k,2)+s(k,1) = k(k-1)!\sum_{i=1}^{k-1} \frac{1}{i}+(k-1)! = k!\sum_{i=1}^{k-1} \frac{1}{i}+(1/k)k! = k!\sum_{i=1}^{k} \frac{1}{i}$. The result now follows for all $m \geq 2$ by the Principle of Mathematical Induction.

Section 5.4

1. Here we find, for example, that
$f(f(a,b),c) = f(a,c) = c$, while
$f(a,f(b,c)) = f(a,b) = a$, so f is *not* associative.

2. (a) For all $a, b \in \mathbf{R}$, $f(a,b) = \lceil a+b \rceil = \lceil b+a \rceil = f(b,a)$, because the real numbers are commutative under addition. Hence f is a commutative (closed) binary operation.
(b) This binary operation is *not* associative. For example,

$$f(f(3.2,4.7),6.4) = f(\lceil 3.2{+}4.7 \rceil, 6.4) = f(\lceil 7.9 \rceil, 6.4) = f(8,6.4) = \lceil 8{+}6.4 \rceil = \lceil 14.4 \rceil = 15,$$

while,

$$f(3.2, f(4.7,6.4)) = f(3.2, \lceil 4.7+6.4 \rceil) = f(3.2, \lceil 11.1 \rceil) = f(3.2, 12) = \lceil 3.2+12 \rceil = \lceil 15.2 \rceil = 16.$$

(c) There is no identity element. If $a \in \mathbf{R} - \mathbf{Z}$ then for any $b \in \mathbf{R}$, $\lceil a+b \rceil \in \mathbf{Z}$. So if x were the identity element we would have $a = f(a,x) = \lceil a+x \rceil$ with $a \in \mathbf{R} - \mathbf{Z}$ and $\lceil a+x \rceil \in \mathbf{Z}$.

3. (a) $f(x,y) = x+y-xy = y+x-yx = f(y,x)$, so the binary operation is commutative.
$f(f(w,x),y) = f(w,x)+y-f(w,x)y = (w+x-wx)+y-(w+x-wx)y = w+x+y-wx-wy-xy+wxy$.
$f(w,f(x,y)) = w+f(x,y)-w\cdot f(x,y) = w+(x+y-xy)-w(x+y-xy) = w+x+y-wx-wy-xy+wxy$.
Since $f(f(w,x),y) = f(w,f(x,y))$, the (closed) binary operation is associative.
(b), (d) Commutative and associative
(c) Neither commutative nor associative.

4. (a) The identity is $z=0$.
(d) The identity is $z=3$.
(b), (c) Neither of these (closed) binary operations has an identity.

5. (a) 25 (b) 5^{25} (c) 5^{25} (d) 5^{10}

6. (a) 5^{24} (b) 5^{15}
(c) $3 \cdot 5^{15}$, because neither a nor b can be an identity.
(d) $3 \cdot 5^{9}$

7. (a) Yes (b) Yes (c) No

8. Each element in A is of the form 2^i for some $1 \le i \le 5$, and $\gcd(2^i, 2^5) = 2^i = \gcd(2^5, 2^i)$, so $2^5 = 32$ is the identity element for f.

9. (a) $|A| = (32)(38) = 1216$.
(b) The identity element for f is $p^{31}q^{37}$.

10. For $n \in \mathbf{Z}^+$ let $p_1, p_2, \ldots, p_n$ be distinct primes and for each $1 \le i \le n$ let M_i be a fixed positive integer. If $A = \{ \prod_{1 \le i \le n} p_i^{e_i} \mid e_i \in \mathbf{N},\ 0 \le e_i \le M_i \}$ define the closed binary operation $f : A \times A \to A$ by $f(a,b) = \gcd(a,b)$.
Then $|A| = \prod_{i=1}^{n}(M_i + 1)$ and the identity element for f is $\prod_{i=1}^{n} p_i^{M_i}$.

11. By the Well-Ordering Principle A has a least element and this same element is the identity for g. If A is finite then A will have a largest element and this same element will be the identity for f. If A is infinite then f cannot have an identity.

12.
(a) $\pi_A(D) = [0, +\infty)$ $\pi_B(D) = \mathbf{R}$
(b) $\pi_A(D) = \mathbf{R}$ $\pi_B(D) = [-1, 1]$
(c) $\pi_A(D) = [-1, 1]$ $\pi_B(D) = [-1, 1]$

13. (a) 5 (b) {(25,25,6), (25,2,4), (60,40,20), (25,40,10)}
(c) A_1, A_2

14. (a) 5
(b) $\{(1, A), (1, D), (1, E), (2, A), (2, D), (2, E)\}$;
$\{(10000, 1, 100), (400, 1, 100), (30, 1, 100), (4000, 1, 250), (400, 1, 250), (15, 1, 250)\}$
(c) $A_1 \times A_2$; $A_2 \times A_5$; $A_3 \times A_5$

Section 5.5

1. Here the socks are the pigeons and the colors are the pigeonholes.

2. The result follows by the Pigeonhole Principle where the eight people are the pigeons and the pigeonholes are the seven days of the week.

3. $26^2 + 1 = 677$

4. Subdivide the set S into the 14 subsets: $\{3\}, \{7, 103\}, \{11, 99\}, \{15, 95\}, \ldots, \{43, 67\}, \{47, 63\}, \{51, 59\}, \{55\}$. By the Pigeonhole Principle if we select at least 15 elements of S then we must have the elements in one of the two-element subsets and these sum to 110.

5. (a) For each $x \in \{1, 2, 3, \ldots, 300\}$ wrote $x = 2^n \cdot m$, where $n \geq 0$ and $\gcd(2, m) = 1$. There are 150 possibilities for m: namely, $1, 3, 5, \ldots, 299$. In selecting 151 numbers from $\{1, 2, 3, \ldots, 300\}$ there must be two numbers of the form $x = 2^s \cdot m$, $y = 2^t \cdot m$. If $x < y$ then $x|y$; otherwise $y < x$ and $y|x$.
(b) If $n + 1$ integers are selected from the set $\{1, 2, 3, \ldots, 2n\}$, then there must be two integers x, y in the selection where $x|y$ or $y|x$.

6. Any selection of size 101 from S must contain two consecutive integers n, $n + 1$ and $\gcd(n, n + 1) = 1$.

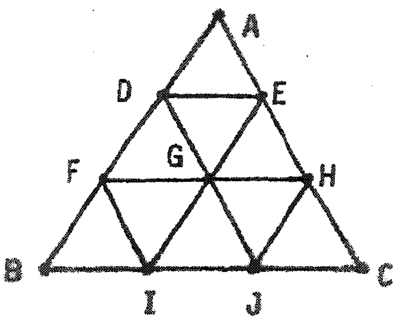
7. (a) Here the pigeons are the integers $1, 2, 3, \ldots, 25$ and the pigeonholes are the 13 sets: $\{1, 25\}, \{2, 24\}, \ldots, \{11, 15\}, \{12, 14\}, \{13\}$. In selecting 14 integers we get the elements in at least one two-element subset, and these sum to 26.
(b) If $S = \{1, 2, 3, \ldots, 2n + 1\}$, for n a positive integer, then any subset of size $n + 2$ from S must contain two elements that sum to $2n + 2$.

8. (a) Since $|S| \geq 3$, $\exists x, y \in S$ where x, y are both even or both odd. In either case $x + y$ is even.
(b) $5 (= 2^2 + 1)$ (c) $9 (= 2^3 + 1)$

(d) For $n \in \mathbf{Z}^+$ let $S = \{(a_1, a_2, \ldots, a_n) | a_i \in \mathbf{Z}^+, 1 \leq i \leq n\}$. If $|S| \geq 2^n + 1$, then S contains two ordered n-tuples $(x_1, x_2, \ldots x_n)$, $(y_1, y_2, \ldots, y_n)$ such that $x_i + y_i$ is even $\forall\, 1 \leq i \leq n$.
(e) 5 – as in part (b).

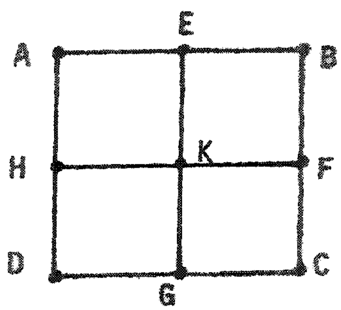
9. (a) For any $t \in \{1, 2, 3, \ldots, 100\}, 1 \leq \sqrt{t} \leq 10$. Selecting 11 elements from $\{1, 2, 3, \ldots, 100\}$ there must be two, say x and y, where $\lfloor \sqrt{x} \rfloor = \lfloor \sqrt{y} \rfloor$, so that $0 < |\sqrt{x} - \sqrt{y}| < 1$.
(b) Let $n \in \mathbf{Z}^+$. If $n+1$ elements are selected from $\{1, 2, 3, \ldots, n^2\}$, then there exist two, say x and y, where $0 < |\sqrt{x} - \sqrt{y}| < 1$.

10.



In triangle ABC, divide each side into three equal parts and form the nine congruent triangles shown in the figure. Let R_1 be the interior of triangle ADE together with the points on segment DE, excluding D,E. Region R_2 is the interior of triangle DFG together with the points on segments DG, FG, excluding D,F. Regions $R_3, \ldots, R_9$ are defined similarly so that the interior of $\triangle$ ABC is the union of these nine regions and $R_i \cap R_j = \emptyset$, for $i \neq j$. Then if 10 points are chosen in the interior of $\triangle$ ABC, at least two of these points are in R_i for some $1 \leq i \leq 9$, and these two points are at a distance less than 1/3 from each other.

11.



Divide the interior of the square into four smaller congruent squares as shown in the figure. Each smaller square has diagonal length $1/\sqrt{2}$. Let region R_1 be the interior of square AEKH together with the points on segment EK, excluding point E. Region R_2 is the interior of square EBFK together with the points on segment FK, excluding points F,K. Regions R_3, R_4 are defined in a similar way. Then if five points are chosen in the interior of square ABCD, at least two are in R_i for some $1 \leq i \leq 4$ and these points are within $1/\sqrt{2}$ (units) of each other.

12. For any five-element subset E of A we find that $1+2+3+4+5 = 15 \leq s_E \leq 115 \leq 21+22+23+24+25$, so there are 116 possible values for such a sum s_E. Since $|A| = 9$, there are $\binom{9}{5} = 126$ five-element subsets of A.

The result now follows by the Pigeonhole Principle where the 126 five-element subsets of A are the pigeons and the 116 possible sums are the pigeonholes.

13. Consider the subsets A of S where $1 \leq |A| \leq 3$. Since $|S| = 5$, there are $\binom{5}{1} + \binom{5}{2} + \binom{5}{3} = 25$ such subsets A. Let s_A denote the sum of the elements in A. Then $1 \leq s_A \leq 7+8+9 = 24$. So by the Pigeonhole Principle, there are two subsets of S whose elements yield the same sum.

14. For $1 \leq i \leq 42$, let x_i count the total number of resumés Brace has sent out from the start of his senior year to the end of the i-th day. Then $1 \leq x_1 < x_2 < \ldots < x_{42} \leq 60$, and $x_1 + 23 < x_2 + 23 < \ldots < x_{42} + 23 \leq 83$. We have 42 distinct numbers $x_1, x_2, \ldots, x_{42}$, and 42 other distinct numbers $x_1 + 23, x_2 + 23, \ldots, x_{42} + 23$, all between 1 and 83 inclusive. By the Pigeonhole Principle $x_i = x_j + 23$ for some $1 \leq j < i \leq 42$; $x_i - x_j = 23$.

15. For $(\emptyset \neq) T \subseteq S$, we have $1 \leq s_T \leq m + (m-1) + \cdots + (m-6) = 7m - 21$. The set S has $2^7 - 1 = 128 - 1 = 127$ nonempty subsets. So by the Pigeonhole Principle we need to have $127 > 7m - 21$ or $148 > 7m$. Hence $7 \leq m \leq 21$.

16. Proof: Consider the $k+1$ integers: (1) 3; (2) 33; (3) 333; ...; and $(k+1)$ $333\ldots3$, where for all $1 \le i \le k+1$, the i-th integer has i digits – each of which is a 3. Since there are $k+1$ integers, it follows from the Division Algorithm and the Pigeonhole Principle that two of these integers, say a and b, have the same remainder when divided by k. Suppose that $a = q_1k + r$, $b = q_2k + r$, and that $a > b$. Then $a - b = (q_1 - q_2)k$, so $k|(a-b)$ and the only digits in $a - b$ are 0's and 3's. [Note: The integer 3 is not special. The result is also true if we replace 3 by any of the digits 1, 2, 4, 5, 6, 7, 8, 9. However, we cannot obtain the result without using the digit 0.]

17. (a) 2,4,1,3
(b) 3,6,9,2,5,8,1,4,7
(c) For $n \ge 2$, there exists a sequence of n^2 distinct real numbers with no decreasing or increasing subsequence of length $n+1$. For example, consider $n, 2n, 3n, \ldots, (n-1)n, n^2, (n-1), (2n-1), \ldots, (n^2-1), (n-2), (2n-2), \ldots, (n^2-2), \ldots, 1, (n+1), (2n+1), \ldots, (n-1)n+1$.
(d) The result in Example 5.49 (for $n \ge 2$) is best possible – in the sense that we cannot reduce the length of the sequence from n^2+1 to n^2 and still obtain the desired subsequence of length $n+1$.

18. This follows from the result due to Paul Erdős and George Szekeres: A sequence of $50(= 7^2+1)$ distinct real numbers contains a decreasing or increasing subsequence of length $8(= 7+1)$.

19. Proof: If not each pigeonhole contains at most k pigeons – for a total of at most kn pigeons. But we have $kn+1$ pigeons. So we have a contradiction and the result then follows.

20. (a) 7 (b) 13 (c) $6(n-1)+1$

21. (a) 1001 (b) 2001
(c) Let $k, n \in \mathbf{Z}^+$. The smallest value for $|S|$ (where $S \subset \mathbf{Z}^+$) so that there exist n elements $x_1, x_2, \ldots, x_n \in S$ where all n of these integers have the same remainder upon division by k is $k(n-1)+1$.

22. Proof: If not, each pigeonhole contains at most $\lfloor (m-1)/n \rfloor$ pigeons – for a total of $n\lfloor (m-1)/n \rfloor \le m-1$ pigeons. But this contradicts the fact that we have m pigeons. The result then follows.
[Note: This result is true even if $m \le n$.]

23. Proof: If not, then the number of pigeons roosting in the first pigeonhole is $x_1 \le p_1 - 1$, the number of pigeons roosting in the second pigeonhole is $x_2 \le p_2 - 1, \ldots$, and the number roosting in the n-th pigeonhole is $x_n \le p_n - 1$. Hence the total number of pigeons is $x_1 + x_2 + \cdots + x_n = (p_1 - 1) + (p_2 - 1) + \cdots + (p_n - 1) = p_1 + p_2 + \cdots + p_n - n < p_1 + p_2 + \cdots + p_n - n + 1$, the number of pigeons we started with. The result now follows because of this contradiction.

24. 42

Section 5.6

1. (a) There are 7! bijective functions on A – of these, 6! satisfy $f(1) = 1$. Hence there are $7! - 6! = 6(6!)$ bijective functions $f : A \to A$ where $f(1) \neq 1$.
(b) $n! - (n-1)! = (n-1)(n-1)!$

2. (a) Here f, g have the same domain A and some codomain $\mathbf{R}$, and for all $x \in A$ we find that

$$g(x) = \frac{2x^2 - 8}{x+2} = \frac{2(x^2-4)}{x+2} = \frac{2(x-2)(x+2)}{(x+2)} = 2(x-2) = 2x - 4 = f(x).$$

Consequently, $f = g$.
(b) Here there is a problem and $f \neq g$. In fact for any nonempty subset A of $\mathbf{R}$, if $-2 \in A$ then g is not defined for A because $g(-2) = 0/0$. [We note that $\frac{x^2-4}{x+2} = x - 2$, for $x \neq -2$.]

3. $9x^2 - 9x + 3 = g(f(x)) = 1 - (ax+b) + (ax+b)^2 = a^2x^2 + (2ab - a)x + (b^2 - b + 1)$. By comparing coefficients on like powers of x, $a = 3, b = -1$ or $a = -3, b = 2$.

4. $g \circ f = \{(1,4), (2,6), (3,10), (4,14)\}$

5. $g^2(A) = g(T \cap (S \cup A)) = T \cap (S \cup [T \cap (S \cup A)]) =$
$T \cap [(S \cup T) \cap (S \cup (S \cup A))] = T \cap [(S \cup T) \cap (S \cup A)] =$
$[T \cap (S \cup T)] \cap (S \cup A) = T \cap (S \cup A) = g(A)$.

6. $(f \circ g)(x) = f(cx + d) = a(cx+d) + b$
$(g \circ f)(x) = g(ax + b) = c(ax + b) + d$
$(f \circ g)(x) = (g \circ f)(x) \iff acx + ad + b = acx + bc + d \iff ad + b = bc + d$

7. (a) $(f \circ g)(x) = 3x - 1$; $(g \circ f)(x) = 3(x-1)$;

$$(g \circ h)(x) = \begin{cases} 0, & x \text{ even}; \\ 3, & x \text{ odd} \end{cases} \qquad (h \circ g)(x) = \begin{cases} 0, & x \text{ even}; \\ 1, & x \text{ odd} \end{cases}$$

$$(f \circ (g \circ h))(x) = f((g \circ h)(x)) = \begin{cases} -1, & x \text{ even}; \\ 2, & x \text{ odd} \end{cases}$$

$$((f \circ g) \circ h)(x) = \begin{cases} (f \circ g)(0), & x \text{ even} \\ (f \circ g)(1), & x \text{ odd} \end{cases} = \begin{cases} -1, & x \text{ even} \\ 2, & x \text{ odd} \end{cases}$$

(b) $f^2(x) = f(f(x)) = x - 2$; $f^3(x) = x - 3$; $g^2(x) = 9x$; $g^3(x) = 27x$; $h^2 = h^3 = h^{500} = h$.

8. (a) If $c \in C$, there is an element $a \in A$ such that $(g \circ f)(a) = c$. Then $g(f(a)) = c$ with $f(a) \in B$, so g is onto.

(b) Let $x, y \in A$. $f(x) = f(y) \Longrightarrow g(f(x)) = g(f(y)) \Longrightarrow (g \circ f)(x) = (g \circ f)(y) \Longrightarrow x = y$, since $g \circ f$ is one-to-one.

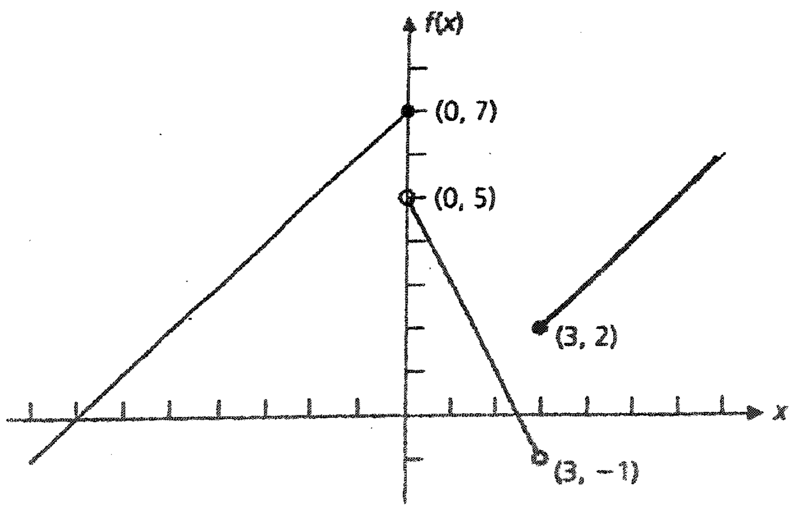
9. (a) $f^{-1}(x) = \frac{1}{2}(\ell n x - 5)$
(b) For $x \in \mathbf{R}^+, (f \circ f^{-1})(x) = f(\frac{1}{2}(\ell n x - 5)) = e^{2((1/2)(\ell n x - 5))+5} = e^{\ell n x - 5 + 5} = e^{\ell n x} = x$; for $x \in \mathbf{R}$, $(f^{-1} \circ f)(x) = f^{-1}(e^{2x+5}) = \frac{1}{2}[\ell n(e^{2x+5}) - 5] = \frac{1}{2}[2x + 5 - 5] = x$.

10. (a) $f^{-1} = \{(x, y)|2y + 3x = 7\}$ (b) $f^{-1} = \{(x, y)|ay + bx = c,\ b \neq 0,\ a \neq 0\}$
(c) $f^{-1} = \{(x, y)|y = x^{1/3}\} = \{(x, y)|x = y^3\}$
(d) Here $f(0) = f(-1) = 0$, so f is not one-to-one, and consequently f is not invertible.

11. f, g invertible $\Longrightarrow$ each of f, g is both one-to-one and onto $\Longrightarrow g \circ f$ is one-to-one and onto $\Longrightarrow g \circ f$ invertible. Since $(g \circ f) \circ (f^{-1} \circ g^{-1}) = 1_C$ and $(f^{-1} \circ g^{-1}) \circ (g \circ f) = 1_A$, $f^{-1} \circ g^{-1}$ is an inverse of $g \circ f$. By uniqueness of inverses $f^{-1} \circ g^{-1} = (g \circ f)^{-1}$.

12. (a) $f^{-1}(\{2\}) = \{a \in A|f(a) \in \{2\}\} = \{a \in A|f(a) = 2\} = \{1\}$
(b) $f^{-1}(\{6\}) = \{a \in A|f(a) \in \{6\}\} = \{a \in A|f(a) = 6\} = \{2, 3, 5\}$
(c) $f^{-1}(\{6, 8\}) = \{a \in A|f(a) \in \{6, 8\}\} = \{a \in A|f(a) = 6 \text{ or } f(a) = 8\} = \{2, 3, 4, 5, 6\}$, because $f(2) = f(3) = f(5) = 6$ and $f(4) = f(6) = 8$.
(d) $f^{-1}(\{6, 8, 10\}) = \{2, 3, 4, 5, 6\} = f^{-1}(\{6, 8\})$ since $f^{-1}(\{10\}) = \emptyset$.
(e) $f^{-1}(\{6, 8, 10, 12\}) = \{2, 3, 4, 5, 6, 7\}$
(f) $f^{-1}(\{10, 12\}) = \{7\}$

13.



(a) $f^{-1}(-10) = \{x \in \mathbf{R} \mid x \leq 10 \text{ and } x + 7 = -10\} = \{-17\}$
$f^{-1}(0) = \{-7, 5/2\}$
$f^{-1}(4) = \{-3, 1/2, 5\}$
$f^{-1}(6) = \{-1, 7\}$
$f^{-1}(7) = \{0, 8\}$
$f^{-1}(8) = \{9\}$

(b) (i) $f^{-1}([-5,-1]) = \{x \in \mathbf{R} \mid x \leq 0 \text{ and } -5 \leq x+7 \leq -1\} \cup \{x \in \mathbf{R} \mid 0 < x < 3 \text{ and } -5 \leq -2x+5 \leq -1\} \cup \{x \in \mathbf{R} \mid 3 \leq x \text{ and } -5 \leq x-1 \leq -1\} = \{x \in \mathbf{R} \mid x \leq 0 \text{ and } -12 \leq x \leq -8\} \cup \{x \in \mathbf{R} \mid 0 < x < 3 \text{ and } 3 \leq x \leq 5\} \cup \{x \in \mathbf{R} \mid 3 \leq x \text{ and } -4 \leq x \leq 0\} = [-12,-8] \cup \emptyset \cup \emptyset = [-12,-8]$
(ii) $f^{-1}([-5,0]) = [-12,-7] \cup [5/2,3)$
(iii) $f^{-1}([-2,4]) = \{x \in \mathbf{R} \mid x \leq 0 \text{ and } -2 \leq x+7 \leq 4\} \cup \{x \in \mathbf{R} \mid 0 < x < 3 \text{ and } -2 \leq -2x+5 \leq 4\} \cup \{x \in \mathbf{R} \mid 3 \leq x \text{ and } -2 \leq x-1 \leq 4\} = \{x \in \mathbf{R} \mid x \leq 0 \text{ and } -9 \leq x \leq -3\} \cup \{x \in \mathbf{R} \mid 0 < x < 3 \text{ and } 1/2 \leq x \leq 7/2\} \cup \{x \in \mathbf{R} \mid 3 \leq x \text{ and } -1 \leq x \leq 5\} = [-9,-3] \cup [1/2,3) \cup [3,5] = [-9,-3] \cup [1/2,5]$
(iv) $f^{-1}((5,10)) = (-2,0] \cup (6,11)$
(v) $f^{-1}([11,17)) = \{x \in \mathbf{R} \mid x \leq 0 \text{ and } 11 \leq x+7 < 17\} \cup \{x \in \mathbf{R} \mid 0 < x < 3 \text{ and } 11 \leq -2x+5 < 17\} \cup \{x \in \mathbf{R} \mid 3 \leq x \text{ and } 11 \leq x-1 < 17\} = \{x \in \mathbf{R} \mid x \leq 0 \text{ and } 4 \leq x < 10\} \cup \{x \in \mathbf{R} \mid 0 < x < 3 \text{ and } -6 < x \leq -3\} \cup \{x \in \mathbf{R} \mid 3 \leq x \text{ and } 12 \leq x < 18\} = \emptyset \cup \emptyset \cup [12,18) = [12,18)$

14. (a) $\{-1,0,1\}$ (b) $\{-1,0,1\}$ (c) $[-1,1]$ (d) $(-1,1)$
(e) $[-2,2]$ (f) $(-3,-2) \cup [-1,0) \cup (0,1] \cup (2,3)$

15. Since $f^{-1}(\{6,7,8\}) = \{1,2\}$ there are three choices for each of $f(1)$ and $f(2)$ – namely, 6, 7 or 8. Furthermore $3,4,5 \notin f^{-1}(\{6,7,8\})$ so $3,4,5 \in f^{-1}(\{9,10,11,12\})$ and we have four choices for each of $f(3)$, $f(4)$, and $f(5)$. Therefore, it follows by the rule of product that there are $3^2 \cdot 4^3 = 576$ functions $f : A \to B$ where $f^{-1}(\{6,7,8\}) = \{1,2\}$.

16.
(a) $[0,2)$ (b) $[-1,2)$ (c) $[0,1)$ (d) $[0,2)$
(e) $[-1,3)$ (f) $[-1,0) \cup [2,4)$

17. (a) The range of $f = \{2,3,4,\ldots\} = \mathbf{Z}^+ - \{1\}$.
(b) Since 1 is not in the range of f the function is not onto.
(c) For all $x, y \in \mathbf{Z}^+$, $f(x) = f(y) \Rightarrow x+1 = y+1 \Rightarrow x = y$, so f is one-to-one.
(d) The range of g is $\mathbf{Z}^+$.
(e) Since $g(\mathbf{Z}^+) = \mathbf{Z}^+$, the codomain of g, this function is onto.
(f) Here $g(1) = 1 = g(2)$, and $1 \neq 2$, so g is not one-to-one.
(g) For all $x \in \mathbf{Z}^+$, $(g \circ f)(x) = g(f(x)) = g(x+1) = \max\{1, (x+1)-1\} = \max\{1, x\} = x$, since $x \in \mathbf{Z}^+$. Hence $g \circ f = 1_{\mathbf{Z}^+}$.
(h) $(f \circ g)(2) = f(\max\{1,1\}) = f(1) = 1+1 = 2$
$(f \circ g)(3) = f(\max\{1,2\}) = f(2) = 2+1 = 3$
$(f \circ g)(4) = f(\max\{1,3\}) = f(3) = 3+1 = 4$
$(f \circ g)(7) = f(\max\{1,6\}) = f(6) = 6+1 = 7$
$(f \circ g)(12) = f(\max\{1,11\}) = f(11) = 11+1 = 12$
$(f \circ g)(25) = f(\max\{1,24\}) = f(24) = 24+1 = 25$
(i) No, because the functions f, g are *not* inverses of each other. The calculations in part (h) may suggest that $f \circ g = 1_{\mathbf{Z}^+}$ since $(f \circ g)(x) = x$ for $x \geq 2$. But we also find that $(f \circ g)(1) = f(\max\{1,0\}) = f(1) = 2$, so $(f \circ g)(1) \neq 1$, and, consequently, $f \circ g \neq 1_{\mathbf{Z}^+}$.

18. (a) $f(\emptyset,\emptyset) = \emptyset = f(\emptyset,\{1\})$ and $(\emptyset,\emptyset) \neq (\emptyset,\{1\})$, so f is not one-to-one.
$g(\{1\},\{2\}) = \{1,2\} = g(\{1,2\},\{2\})$ and $(\{1\},\{2\}) \neq (\{1,2\},\{2\})$, so g is not one-to-one.
$h(\{1\},\{2\}) = \{1,2\} = h(\{2\},\{1\})$ and $(\{1\},\{2\}) \neq (\{2\},\{1\})$, so h is not one-to-one.
(b) For each subset A of $\mathbf{Z}^+$, $f(A,A) = g(A,A) = h(A,\emptyset) = A$, so each of the three functions f, g, and h, is an onto function.
(c) From the results in part (a) it follows that none of these functions is invertible.
(d) The sets $f^{-1}(\emptyset)$, $h^{-1}(\emptyset)$, $f^{-1}(\{1\})$, $h^{-1}(\{3\})$, $f^{-1}(\{4,7\})$, and $h^{-1}(\{5,9\})$, are all infinite.
(e) $|g^{-1}(\emptyset) = \{(\emptyset,\emptyset)\}$, so $|g^{-1}(\emptyset)| = 1$.
$g^{-1}(\{2\}) = \{(\emptyset,\{2\}), (\{2\},\emptyset), (\{2\},\{2\})\}$, so $|g^{-1}(\{2\})| = 3$
$|g^{-1}(\{8,12\})| = 9$.

19. (a) $a \in f^{-1}(B_1 \cap B_2) \iff f(a) \in B_1 \cap B_2 \iff f(a) \in B_1$ and $f(a) \in B_2 \iff a \in f^{-1}(B_1)$ and $a \in f^{-1}(B_2) \iff a \in f^{-1}(B_1) \cap f^{-1}(B_2)$
(c) $a \in f^{-1}(\overline{B_1}) \iff f(a) \in \overline{B_1} \iff f(a) \notin B_1 \iff a \notin f^{-1}(B_1) \iff a \in \overline{f^{-1}(B_1)}$

20. (a) (i) $f(x) = 2x$; (ii) $f(x) = \lfloor x/2 \rfloor$
(b) No. The set $\mathbf{Z}$ is not finite.

21. (a) Suppose that $x_1, x_2 \in \mathbf{Z}$ and $f(x_1) = f(x_2)$. Then either $f(x_1)$, $f(x_2)$ are both even or they are both odd. If they are both even, then $f(x_1) = f(x_2) \Rightarrow -2x_1 = -2x_2 \Rightarrow x_1 = x_2$. Otherwise, $f(x_1)$, $f(x_2)$ are both odd and $f(x_1) = f(x_2) \Rightarrow 2x_1 - 1 = 2x_2 - 1 \Rightarrow 2x_1 = 2x_2 \Rightarrow x_1 = x_2$. Consequently, the function f is one-to-one.

In order to prove that f is an onto function let $n \in \mathbf{N}$. If n is even, then $(-n/2) \in \mathbf{Z}$ and $(-n/2) < 0$, and $f(-n/2) = -2(-n/2) = n$. For the case where n is odd we find that $(n+1)/2 \in \mathbf{Z}$ and $(n+1)/2 > 0$, and $f((n+1)/2) = 2[(n+1)/2] - 1 = (n+1) - 1 = n$. Hence f is onto.

(b) $f^{-1} : \mathbf{N} \to \mathbf{Z}$, where

$$f^{-1}(x) = \begin{cases} (\frac{1}{2})(x+1), & x = 1,3,5,7\ldots \\ -x/2, & x = 0,2,4,6,\ldots \end{cases}$$

22. It follows from Theorem 5.11 that there are 5! invertible functions $f : A \longrightarrow B$.

23. (a) For all $n \in \mathbf{N}$, $(g \circ f)(n) = (h \circ f)(n) = (k \circ f)(n) = n$.
(b) The results in part (a) do not contradict Theorem 5.7. For although $g \circ f = h \circ f = k \circ f = 1_{\mathbf{N}}$, we note that
(i) $(f \circ g)(1) = f(\lfloor 1/3 \rfloor) = f(0) = 3 \cdot 0 = 0 \neq 1$, so $f \circ g \neq 1_{\mathbf{N}}$;
(ii) $(f \circ h)(1) = f(\lfloor 2/3 \rfloor) = f(0) = 3 \cdot 0 = 0 \neq 1$, so $f \circ h \neq 1_{\mathbf{N}}$; and
(iii) $(f \circ k)(1) = f(\lfloor 3/3 \rfloor) = f(1) = 3 \cdot 1 = 3 \neq 1$, so $f \circ k \neq 1_{\mathbf{N}}$.
Consequently, none of g, h, and k, is the inverse of f. (After all, since f is *not* onto it is *not* invertible.)

Section 5.7

1.

(a) $f \in O(n)$	(b) $f \in O(1)$	(c) $f \in O(n^3)$
(d) $f \in O(n^2)$	(e) $f \in O(n^3)$	(f) $f \in O(n^2)$
(g) $f \in O(n^2)$		

2. Let $m = 1$ and $k = 1$ in Definition 5.23. Then $\forall n \geq k\ |f(n)| = n < n + (1/n) = |g(n)|$, so $f \in O(g)$.
Now let $m = 2$ and $k = 1$. Then $\forall n \geq k\ |g(n)| = n + (1/n) \leq n + n = 2n = 2|f(n)|$, and $g \in O(f)$.

3. (a) For all $n \in \mathbf{Z}^+, 0 \leq \log_2 n < n$. So let $k = 1$ and $m = 200$ in Definition 5.23. Then $|f(n)| = 100 \log_2 n = 100((1/2)\ \log_2 n) < 200((1/2)n) = 200|g(n)|$, so $f \in O(g)$.
(b) For $n = 6$, $2^n = 64 < 3096 = 4096 - 1000 = 2^{12} - 1000 = 2^{2n} - 1000$. Assuming that $2^k < 2^{2k} - 1000$ for $n = k \geq 6$, we find that $2 < 2^2 \Longrightarrow 2(2^k) < 2^2(2^{2k} - 1000) < 2^2 2^{2k} - 1000$, or $2^{k+1} < 2^{2(k+1)} - 1000$, so $f(n) < g(n)$ for all $n \geq 6$. Therefore, with $k = 6$ and $m = 1$ in Definition 5.23 we find that for $n \geq k\ |f(n)| \leq m|g(n)|$ and $f \in O(g)$.
(c) For all $n \geq 4$, $n^2 \leq 2^n$ (A formal proof of this can be given by mathematical induction.) So let $k = 4$ and $m = 3$ in Definition 5.23. Then for $n \geq k$, $|f(n)| = 3n^2 \leq 3(2^n) < 3(2^n + 2n) = m|g(n)|$ and $f \in O(g)$.

4. Let $m = 11$ and $k = 1$. Then $\forall n \geq k\ |f(n)| = n + 100 \leq 11n^2 = m|g(n)|$, so $f \in O(g)$. However, $\forall m \in \mathbf{R}^+\ \forall k \in \mathbf{Z}^+$ choose $n > max\{k, 100 + m\}$. Then $n^2 > (100 + m)n = 100n + mn > 100m + mn = m(100 + n) = m|f(n)|$, so $g \notin O(f)$.

5. To show that $f \in O(g)$, let $k = 1$ and $m = 4$ in Definition 5.23. Then for all $n \geq k$, $|f(n)| = n^2 + n \leq n^2 + n^2 = 2n^2 \leq 2n^3 = 4((1/2)(n^3)) = 4|g(n)|$, and f is dominated by g.
To show that $g \notin O(f)$, we follow the idea given in Example 5.66 – namely that

$$\forall m \in \mathbf{R}^+\ \forall k \in \mathbf{Z}^+\ \exists n \in \mathbf{Z}^+\ [(n \geq k) \wedge (|g(n)| > m|f(n)|)].$$

So not matter what the values of m and k are, choose $n > max\{4m, k\}$. Then $|g(n)| = (1/2)n^3 > (1/2)(4m)n^2 = m(2n^2) \geq m(n^2 + n) = m|f(n)|$, so $g \notin O(f)$.

6. $\forall m \in \mathbf{R}^+\ \forall k \in \mathbf{Z}^+$ choose $n > max\{k, m\}$ with n odd. Then $n = |f(n)| > m = m \cdot 1 = m|g(n)|$, so $f \notin O(g)$. In a similar way, $\forall m \in \mathbf{R}^+\ \forall k \in \mathbf{Z}^+$ now choose $n > max\{k, m\}$ with n even. Then $n = |g(n)| > m = m \cdot 1 = m|f(n)|$, and $g \notin O(f)$.

7. For all $n \geq 1, \log_2 n \leq n$, so with $k = 1$ and $m = 1$ in Definition 5.23 we have $|g(n)| = \log_2 n \leq n = m \cdot n = m|f(n)|$. Hence $g \in O(f)$.
To show that $f \in O(g)$ we first observe that $\lim_{n \to \infty} \frac{n}{\log_2 n} = +\infty$. (This can be established by using L'Hospital's Rule from the Calculus.) Since $\lim_{n \to \infty} \frac{n}{\log_2 n} = +\infty$ we

find that for every $m \in \mathbf{R}^+$ and $k \in \mathbf{Z}^+$ there is an $n \in \mathbf{Z}^+$ such that $\frac{n}{\log_2 n} > m$, or $|f(n)| = n > m \log_2 n = m|g(n)|$. Hence $f \notin O(g)$.

8. $f \in O(g) \Longrightarrow \exists m_1 \in \mathbf{R}^+ \; \exists k_1 \in \mathbf{Z}^+$ so that $\forall n \geq k_1 \; |f(n)| \leq m_1|g(n)|$. $g \in O(h) \Longrightarrow \exists m_2 \in \mathbf{R}^+ \; \exists k_2 \in \mathbf{Z}^+$ so that $\forall n \geq k_2 \; |g(n)| \leq m_2|h(n)|$. Therefore, $\forall n \geq max\{k_1, k_2\}$ we have $|f(n)| \leq m_1|g(n)| \leq m_1 m_2|h(n)|$ and $f \in O(h)$.

9. Since $f \in O(g)$, there exists $m \in \mathbf{R}^+, k \in \mathbf{Z}^+$ so that $|f(n)| \leq m|g(n)|$ for all $n \geq k$. But then $|f(n)| \leq [m/|c|]|cg(n)|$ for all $n \geq k$, so $f \in O(cg)$.

10. (a) Let $k = 1$ and $m = 1$ in Definition 5.23.
(b) If $h \in O(f)$ and $f \in O(g)$, then $h \in O(g)$ by Exercise 8. Likewise, if $h \in O(g)$ and $g \in O(f)$ then $h \in O(f)$ – again by Exercise 8.
(c) This follows from parts (a) and (b).

11. (a) For all $n \geq 1$, $f(n) = 5n^2 + 3n > n^2 = g(n)$. So with $M = 1$ and $k = 1$, we have $|f(n)| \geq M|g(n)|$ for all $n \geq k$ and it follows that $f \in \Omega(g)$.

(b) For all $n \geq 1$, $g(n) = n^2 = (1/10)(5n^2 + 5n^2) > (1/10)(5n^2 + 3n) = (1/10)f(n)$. So with $M = (1/10)$ and $k = 1$, we find that $|g(n)| \geq M|f(n)|$ for all $n \geq k$ and it follows that $g \in \Omega(f)$.

(c) For all $n \geq 1$, $f(n) = 5n^2 + 3n > n = h(n)$. With $M = 1$ and $k = 1$, we have $|f(n)| \geq M|h(n)|$ for all $n \geq k$ and so $f \in \Omega(h)$.

(d) Suppose that $h \in \Omega(f)$. If so, there exist $M \in \mathbf{R}^+$ and $k \in \mathbf{Z}^+$ with $n = |h(n)| \geq M|f(n)| = M(5n^2+3n)$ for all $n \geq k$. Then $0 < M \leq n/(5n^2+3n) = 1/(5n+3)$. But how can M be a positive *constant* while $1/(5n+3)$ approaches 0 as n (a *variable*) gets larger? From this contradiction it follows that $h \notin \Omega(f)$.

12. Proof: Suppose that $f \in \Omega(g)$. Then there exist $M \in \mathbf{R}^+$ and $k \in \mathbf{Z}^+$ such that $|f(n)| > M|g(n)|$ for all $n \geq k$. Consequently, $|g(n)| \leq (1/M)|f(n)|$ for all $n \geq k$, so $g \in O(f)$.

Conversely, $g \in O(f) \Rightarrow \exists m \in \mathbf{R}^+ \; \exists k \in \mathbf{Z}^+ \; \forall n \geq k \; (|g(n)| \leq m|f(n)|) \Rightarrow \exists m \in \mathbf{R}^+ \; \exists k \in \mathbf{Z}^+ \; \forall n \geq k \; (|f(n)| \geq (1/m)|g(n)|) \Rightarrow \exists M \in \mathbf{R}^+ \; \exists k \in \mathbf{Z}^+ \; \forall n \geq k \; (|f(n)| \geq M|g(n)|) \Rightarrow f \in \Omega(g)$. [Here $M = 1/m$.] [Note: Upon replacing each occurrence of $\Rightarrow$ by $\Leftrightarrow$ we can establish this "if and only if" proof without the first (separate) part in the first paragraph.]

13. (a) For $n \geq 1$, $f(n) = \sum_{i=1}^n i = n(n+1)/2 = (n^2/2) + (n/2) > (n^2/2)$. With $k = 1$ and $M = 1/2$, we have $|f(n)| \geq M|n^2|$ for all $n \geq k$. Hence $f \in \Omega(n^2)$.
(b) $\sum_{i=1}^n i^2 = 1^2 + 2^2 + \cdots + n^2 > \lceil n/2 \rceil^2 + \cdots + n^2 > \lceil n/2 \rceil^2 + \cdots + \lceil n/2 \rceil^2 = \lceil (n+1)/2 \rceil \lceil n/2 \rceil^2 > n^3/8$. With $k = 1$ and $M = 1/8$, we have $|g(n)| \geq M|n^3|$ for all $n \geq k$. Hence $g \in \Omega(n^3)$.

Alternately, for $n \geq 1$, $g(n) = \sum_{i=1}^n i^2 = n(n+1)(2n+1)/6 = (2n^3 + 3n^2 + n)/6 > n^3/6$.

With $k = 1$ and $M = 1/6$, we find that $|g(n)| \geq M|n^3|$ for all $n \geq k$ – so $g \in \Omega(n^3)$.
(c) $\sum_{i=1}^{n} i^t = 1^t+2^t+\cdots+n^t > \lceil n/2 \rceil^t+\cdots+n^t > \lceil n/2 \rceil^t+\cdots+\lceil n/2 \rceil^t = \lceil (n+1)/2 \rceil \lceil n/2 \rceil^t > (n/2)^{t+1}$. With $k = 1$ and $M = (1/2)^{t+1}$, we have $|h(n)| \geq M|n^{t+1}|$ for all $n \geq k$. Hence $h \in \Omega(n^{t+1})$.

14. Proof: $f \in \Theta(g) \Rightarrow \exists m_1, m_2 \in \mathbf{R}^+ \; \exists k \in \mathbf{Z}^+ \; \forall n \geq k \; m_1|g(n)| \leq |f(n)| \leq m_2|g(n)| \Rightarrow \exists m_1 \in \mathbf{R}^+ \; \exists k \in \mathbf{Z}^+ \; \forall n \leq k \; m_1|g(n)| \leq |f(n)|$ and $\exists m_2 \in \mathbf{R}^+ \; \exists k \in \mathbf{Z}^+ \; \forall n \geq k \; |f(n)| \leq m_2|g(n)| \Rightarrow f \in \Omega(g)$ and $f \in O(g)$.

Conversely, $f \in \Omega(g) \Rightarrow \exists m_1 \in \mathbf{R}^+ \; \exists k_1 \in \mathbf{Z}^+ \; \forall n \geq k_1 \; m_1|g(n)| \leq |f(n)|$. Likewise, $f \in O(g) \Rightarrow \exists m_2 \in \mathbf{R}^+ \; \exists k_2 \in \mathbf{Z}^+ \; \forall n \geq k_2 \; |f(n)| \leq m_2|g(n)|$. Let $k = \max\{k_1, k_2\}$. Then for all $n \geq k$, $m_1|g(n)| \leq |f(n)| \leq m_2|g(n)|$, so $f \in \Theta(g)$.

15. Proof: $f \in \Theta(g) \Rightarrow f \in \Omega(g)$ and $f \in O(g)$ (from Exercise 14 of this section) $\Rightarrow g \in O(f)$ and $g \in \Omega(f)$ (from Exercise 12 of this section) $\Rightarrow g \in \Theta(f)$.

16. Proof: Part (a) follows from Exercises 14 and 13(a) of this section and part (a) of Example 5.68.

The situation is similar for parts (b) and (c).

Section 5.8

1.
(a) $f \in O(n^2)$ (b) $f \in O(n^3)$ (c) $f \in O(n^2)$
(d) $f \in O(\log_2 n)$ (e) $f \in O(n \log_2 n)$

2. (a) $f \in O(n)$ (b) $f \in O(n)$

3. (a) For the following program segment the value of the integer n, and the values of the array entries $A[1], A[2], A[3], \ldots, A[n]$ are supplied beforehand. Also, the variables i, Max, and Location that are used here are integer variables.

```
Begin
    Max := A[1];
    Location := 1;
    If n = 1 then
        Begin
            Writeln ('The first occurrence of the maximum ');
            Write ('entry in the array is at position 1.')
        End;
    If n > 1 then
        Begin
            For i := 2 to n do
                If Max < A[i] then
                    Begin
```

```
                Max := A[i];
                Location := i
            End;
        Writeln (' The first occurrence of the maximum ');
        Write (' entry in the array is at position ', i:0, '.')
    End
End;
```

(b) If, as in Exercise 2, we define the worst-case complexity function $f(n)$ as the number of times the comparison Max $< A[i]$ is executed, then $f(n) = n - 1$ for all $n \in \mathbf{Z}^+$, and $f \in \mathcal{O}(n)$.

4. (a) For the following program segment the value of the integer n, and the values of the array entries $A[1], A[2], A[3], \ldots, A[n]$ are supplied earlier in the program. Also the variables i, Max, and Min that are used here are integer variables.

```
Begin
    Min := A[1];
    Max := A[1];
    For i := 2 to n do
        Begin
            If A[i] < Min then
                Min := A[i];
            If A[i] > Max then
                Max := A[i];
        End;
    Writeln (' The minimum value in the array is ', Min :0);
    Write (' and the maximum value is ', Max:0, '.')
End;
```

(b) Here we define the worst-case time-complexity function $f(n)$ as the number of comparisons that are executed in the For loop. Consequently, $f(n) = 2(n-1)$ for all $n \in \mathbf{Z}^+$ and $f \in \mathcal{O}(n)$.

5. (a) Here there are five additions and ten multiplications.
(b) For the general case there are n additions and $2n$ multiplications.

6. (a) For each iteration of the for loop there is one addition and one multiplication. Therefore, in total, there are five additions and five multiplications.
(b) For the general case there are n additions and n multiplications.

7. Proof: For $n = 1$, we find that $a_1 = 0 = \lfloor 0 \rfloor = \lfloor \log_2 1 \rfloor$, so the result is true in this first case.
Now assume the result true for all $n = 1, 2, 3, \ldots, k$, where $k \geq 1$, and consider the cases for $n = k + 1$.

(i) $n = k+1 = 2^m$, where $m \in \mathbf{Z}^+$: Here $a_n = 1 + a_{\lfloor n/2 \rfloor} = 1 + a_{2^{m-1}} = 1 + \lfloor \log_2 2^{m-1} \rfloor = 1 + (m-1) = m = \lfloor \log_2 2^m \rfloor = \lfloor \log_2 n \rfloor$; and

(ii) $n = k+1 = 2^m + r$, where $m \in \mathbf{Z}^+$ and $0 < r < 2^m$: Here $2^m < n < 2^{m+1}$, so we have

(1) $2^{m-1} < (n/2) < 2^m$;

(2) $2^{m-1} = \lfloor 2^{m-1} \rfloor \leq \lfloor n/2 \rfloor < \lfloor 2^m \rfloor = 2^m$; and

(3) $m - 1 = \log_2 2^{m-1} \leq \log_2 \lfloor n/2 \rfloor < \log_2 2^m = m$.

Consequently, $\lfloor \log_2 \lfloor n/2 \rfloor \rfloor = m - 1$ and $a_n = 1 + a_{\lfloor n/2 \rfloor} = 1 + \lfloor \log_2 \lfloor n/2 \rfloor \rfloor = 1 + (m-1) = m = \lfloor \log_2 n \rfloor$.

Therefore it follows from the Alternative Form of the Principle of Mathematical Induction that $a_n = \lfloor \log_2 n \rfloor$ for all $n \in \mathbf{Z}^+$.

8. We claim that $a_n = \lceil \log_2 n \rceil$ for all $n \in \mathbf{Z}^+$.
Proof: When $n = 1$ we have $a_1 = 0 = \lceil 0 \rceil = \lceil \log_2 1 \rceil$, and this establishes our basis step. For the inductive step we assume the result true for all $n = 1, 2, 3, \ldots, k$ (≥ 1) and consider what happens at $n = k+1$.

(i) $n = k+1 = 2^m$, where $m \in \mathbf{Z}^+$: Here $a_n = 1 + a_{\lceil n/2 \rceil} = 1 + a_{2^{m-1}} = 1 + \lceil \log_2 2^{m-1} \rceil = 1 + (m-1) = m = \lceil \log_2 2^m \rceil = \lceil \log_2 n \rceil$.

(ii) $n = k+1 = 2^m + r$, where $m \in \mathbf{Z}^+$ and $0 < r < 2^m$: Here $2^m < n < 2^{m+1}$ and we find that

(1) $2^{m-1} < n/2 < 2^m$;

(2) $2^{m-1} = \lceil 2^{m-1} \rceil < \lceil n/2 \rceil \leq \lceil 2^m \rceil = 2^m$; and

(3) $m - 1 = \log_2 2^{m-1} < \log_2 \lceil n/2 \rceil \leq \log_2 2^m = m$.

Therefore, $\lceil \log_2 \lceil n/2 \rceil \rceil = m$ and $a_n = 1 + a_{\lceil n/2 \rceil} = 1 + \lceil \log_2 \lceil n/2 \rceil \rceil = 1 + m = \lceil \log_2 n \rceil$, since $2^m < n < 2^{m+1} \Rightarrow \log_2 2^m = m < \log_2 n < m + 1 = \log_2 2^{m+1} \Rightarrow m < \lceil \log_2 n \rceil = m + 1$.

Consequently, it follows from the Alternative Form of the Principle of Mathematical Induction that $a_n = \lceil \log_2 n \rceil$ for all $n \in \mathbf{Z}^+$.

9. Here $np = 3/4$ and $q = 1 - np = 1/4$, so $E(X) = np(n+1)/2 + nq = (3/4)[(n+1)/2] + (1/4)n = (3/8)n + (3/8) + (1/4)n = (5/8)n + (3/8)$.

10. $Pr(X = i) = i/[n(n+1)]$, so $\sum_{i=1}^{n} Pr(X = i) = \sum_{i=1}^{n} i/[n(n+1)] = (1/[n(n+1)]) \sum_{i=1}^{n} i = (1/[n(n+1)])[n(n+1)/2] = 1/2$ and $q = 1 - (1/2) = 1/2$.
$E(X) = \sum_{i=1}^{n} i^2/[n(n+1)] + (1/2)n = [1/[n(n+1)]] \sum_{i=1}^{n} i^2 + (1/2)n = \frac{1}{n(n+1)} \cdot \frac{n(n+1)(2n+1)}{6} + \frac{n}{2} = \frac{2n+1}{6} + \frac{n}{2} = \frac{5n}{6} + \frac{1}{6}$.

11.

a)
```
procedure LocateRepeat (n: positive integer; a1, a2, a3, ..., an: integers)
begin
    location := 0
    i := 2
    while i ≤ n and location = 0 do
        begin
            j := 1
            while j < i and location = 0 do
```

```
                if a_j = a_i then location := i
                else j := j + 1
            i := i + 1
        end
end {location is the subscript of the first array entry that repeats a previous
    array entry; location is 0 if the array contains n distinct integers}
```

b) For $n \geq 2$, let $f(n)$ count the maximum number of times the second **while** loop is executed. The second **while** loop is executed at most $n-1$ times for each value of i, where $2 \leq i \leq n$. Consequently, $f(n) = 1 + 2 + 3 + \cdots + (n-1) = (n-1)(n)/2$, which occurs when the array consists of n distinct integers or when the only repeat is a_{n-1} and a_n. Since $(n-1)(n)/2 = (1/2)(n^2 - n)$ we have $f \in O(n^2)$.

12.

a)

```
procedure FirstDecrease (n: positive integer; a_1, a_2, a_3, ..., a_n: integers)
begin
    location := 0
    i := 2
    while i ≤ n and location = 0 do
            if a_i < a_{i-1} then location := i
            else i := i + 1
end {location is the subscript of the first array entry that is smaller than its
    immediate predecessor; location is 0 if the n integers in the array
    are in increasing order}
```

b) For $n \geq 2$, let $f(n)$ count the maximum number of comparisons made in the **while** loop. This is $n - 1$, which occurs if the integers in the array are in ascending order or if $a_1 < a_2 < a_3 < \ldots < a_{n-1}$ and $a_n < a_{n-1}$. Consequently, $f \in O(n)$.

Supplementary Exercises

1. (a) If either A or B is $\emptyset$ then $A \times B = \emptyset = A \cap B$ and the result is true.
For A, B nonempty we find that:

$(x, y) \in (A \times B) \cap (B \times A) \Rightarrow (x, y) \in A \times B$ and $(x, y) \in B \times A \Rightarrow (x \in A$ and $y \in B)$ and $(x \in B$ and $y \in A) \Rightarrow x \in A \cap B$ and $y \in A \cap B \Rightarrow (x, y) \in (A \cap B) \times (A \cap B)$; and

$(x, y) \in (A \cap B) \times (A \cap B) \Rightarrow (x \in A$ and $x \in B)$ and $(y \in A$ and $y \in B) \Rightarrow (x, y) \in A \times B$ and $(x, y) \in B \times A \Rightarrow (x, y) \in (A \times B) \cap (B \times A)$.

Consequently, $(A \times B) \cap (B \times A) = (A \cap B) \times (A \cap B)$.

(b) If either A or B is $\emptyset$ then $A \times B = \emptyset = B \times A$ and the result follows.
If not, let $(x, y) \in (A \times B) \cup (B \times A)$. Then

$(x,y) \in (A \times B) \cup (B \times A) \Rightarrow (x,y) \in A \times B$ or $(x,y) \in (B \times A) \Rightarrow (x \in A$ and $y \in B)$ or $(x \in B$ and $y \in A) \Rightarrow (x \in A$ or $x \in B)$ and $(y \in A$ or $y \in B) \Rightarrow x, y \in A \cup B \Rightarrow (x,y) \in (A \cup B) \times (A \cup B)$.

2. (a) True (b) False: Let $A = \{1,2\}, B = \{x,y\}, f = \{(1,x),(2,y)\}$.
(c) False: Let $f : \mathbf{Z} \to \mathbf{Z}, f(x) = 2x$. (d) True.
(e) False: Let $A = \{1,2\}, B = \{1,2,3\}, C = \{1,2,3,4\}, f = \{(1,1),(2,2)\}$, $g = \{(1,1),(2,2),(3,3)\}, h = \{(1,1),(2,2),(3,4)\}$.
(f) False. Let $A = \{1,2,3,4\}, B = \{5,6\}, A_1 = \{1,2\}, A_2 = \{2,3,4\}$, $f = \{(1,5),(2,6),(3,5),(4,5)\}$. Then $f(A_1 \cap A_2) = f(2) = \{6\}$, but $f(A_1) \cap f(A_2) = \{5,6\}$.
(g) True

3. (a) $f(1) = f(1 \cdot 1) = 1 \cdot f(1) + 1 \cdot f(1)$, so $f(1) = 0$.
(b) $f(0) = 0$
(c) Proof (by Mathematical Induction): When $a = 0$ the result is true, so consider $a \neq 0$. For $n = 1$, $f(a^n) = f(a) = 1 \cdot a^0 \cdot f(a) = na^{n-1}f(a)$, so the result follows in this first case, and this establishes our basis step. Assume the result true for $n = k (\geq 1)$ – that is, $f(a^k) = ka^{k-1}f(a)$. For $n = k+1$ we have $f(a^{k+1}) = f(a \cdot a^k) = af(a^k) + a^k f(a) = aka^{k-1}f(a) + a^k f(a) = ka^k f(a) + a^k f(a) = (k+1)a^k f(a)$. Consequently, the truth of the result for $n = k+1$ follows from the truth of the result for $n = k$. So by the Principle of Mathematical Induction the result is true for all $n \in \mathbf{Z}^+$.

4. $2^{|A \times B|} = 262,144 \Longrightarrow |A \times B| = 18 \Longrightarrow |A| = 2, |B| = 9$ or $|A| = 3, |B| = 6$.

5. $(x,y) \in (A \cap B) \times (C \cap D) \Longleftrightarrow x \in A \cap B, y \in C \cap D \Longleftrightarrow (x \in A, y \in C)$ and $(x \in B, y \in D) \Longrightarrow (x,y) \in A \times C$ and $(x,y) \in B \times D \Longleftrightarrow (x,y) \in (A \times C) \cap (B \times D)$

6. (a) 5! (b) 4!

7. If $0 \leq x < 1$, then $\lfloor x \rfloor = 0$ and $x^2 = 1/2$. So $x = 1/\sqrt{2}$.
If $1 \leq x < 2$, then $\lfloor x \rfloor = 1$ and $x^2 = 3/2$. So $x = \sqrt{3/2}$.
For $k \in \mathbf{Z}^+$ and $k \geq 2$, if $k \leq x < k+1$, then $\lfloor x \rfloor = k$ and if x satisfies the given equation we have $x^2 = k + (1/2)$. But for $k \geq 2$ we find that $k(k-1) > 0$, so $k(k-1) \geq 1 > 1/2$, and $k^2 - k > 1/2$. Now $k^2 > k + (1/2) \Rightarrow k > \sqrt{k + (1/2)} = x$ and we do *not* have $k \leq x < k+1$.
Finally, let $k \in \mathbf{Z}^+$ and consider $-k \leq x < -k+1$. Then $x^2 - \lfloor x \rfloor = x^2 - (-k) = x^2 + k$, and $x^2 - \lfloor x \rfloor = 1/2 \Rightarrow x^2 = -k + 1/2 < 0$, so x cannot be a real number.
Consequently, there are only two real numbers that satisfy the equation $x^2 - \lfloor x \rfloor = 1/2$ — namely, $x = 1/\sqrt{2}$ and $x = \sqrt{3/2}$.

8. Proof: First we show that the result holds for the first part of the recursive definition. Since $2 \cdot 1 = 2 \geq 1$ we find the result true in part (1). In order to complete the proof we need to verify that every ordered pair (s,t) in $\mathcal{R}$ that results from part (2) of the definition satisfies the condition $2s \geq t$. We consider three cases:

(i) $(a+1, b)$ with $(a, b) \in \mathcal{R}$: Here we have $2a \geq b$, and since $a+1 \geq a$ it follows that $2(a+1) \geq 2a \geq b$;
(ii) $(a+1, b+1)$ with $(a, b) \in \mathcal{R}$: Now we find that $2a \geq b \Rightarrow 2a+2 \geq b+1 \Rightarrow 2(a+1) \geq b+1$; and
(iii) $(a+1, b+2)$ with $(a, b) \in \mathcal{R}$: In this last case it follows that $2a \geq b \Rightarrow 2a+2 \geq b+2 \Rightarrow 2(a+1) \geq b+2$.
Consequently, for all $(a, b) \in \mathcal{R}$ we have $2a \geq b$.

9. (a) $f^2(x) = f(f(x)) = a(f(x)+b) - b = a[(a(x+b)-b)+b] - b = a^2(x+b) - b$
$f^3(x) = f(f^2(x)) = f(a^2(x+b)-b) = a[(a^2(x+b)-b)+b] - b = a^3(x+b) - b$

(b) Conjecture: For $n \in \mathbf{Z}^+$, $f^n(x) = a^n(x+b) - b$. Proof (by Mathematical Induction): The formula is true for $n = 1$ – by the definition of $f(x)$. Hence we have our basis step. Assume the formula true for $n = k(\geq 1)$ – that is, $f^k(x) = a^k(x+b) - b$. Now consider $n = k+1$. We find that $f^{k+1}(x) = f(f^k(x)) = f(a^k(x+b)-b) = a[(a^k(x+b)-b)+b] - b = a^{k+1}(x+b) - b$. Since the truth of the formula at $n = k$ implies the truth of the formula at $n = k+1$, it follows that the formula is valid for all $n \in \mathbf{Z}^+$ – by the Principle of Mathematical Induction.

10. Let $n = |A| - |A_1|$. Since $|B|^n$ is the number of ways to extend f to A and $|B|^n = 6^n = 216$, then $n = 3$ and $|A| = 8$.

11. (a) $(7 \times 6 \times 5 \times 4 \times 3)/(7^5) \doteq 0.15$.
(b) For the computer program the elements of B are replaced by {1,2,3,4,5,6,7}.

```
10   Random
20   Dim F(5)
30   For I = 1 To 5
40        F(I) = Int(Rnd*7 + 1)
50   Next I
60   For J = 2 To 5
70        For K = 1 To J - 1
80             If F(J) = F(K) then GOTO 120
90        Next K
100  Next J
110  GOTO 140
120  C = C + 1
130  GOTO 10
140  C = C + 1
150  Print "After "; C; " generations the resulting"
160  Print "function is one-to-one."
170  Print "The one-to-one function is given as:"
180  For I = 1 To 5
190       Print "("; I; ","; F(I); ")"
```

```
200  Next I
210  End
```

12. For each subset A of S, let s_A denote the sum of the elements of A. Consider only those nonempty subsets A of S where $|A| \leq 5$. There are $2^7 - 1 - 1 - 7 = 119$ such subsets and here $1 \leq s_A \leq 20+21+22+23+24 = 110$. The result follows by the Pigeonhole Principle for there are 119 subsets (pigeons) and 110 possible sums (pigeonholes).

13. For $1 \leq i \leq 10$, let x_i be the number of letters typed on day i. Then $x_1 + x_2 + x_3 + \ldots + x_8 + x_9 + x_{10} = 84$, or $x_3 + \ldots + x_8 = 54$. Suppose that $x_1 + x_2 + x_3 < 25, x_2 + x_3 + x_4 < 25, \ldots, x_8 + x_9 + x_{10} < 25$. Then $x_1 + 2x_2 + 3(x_3 + \ldots + x_8) + 2x_9 + x_{10} < 8(25) = 200$, or $3(x_3 + \ldots + x_8) < 160$. Consequently, $54 = x_3 + \ldots + x_8 < (160)/3 = 53\ 1/3$.

14. If two elements in $\{x_1, x_2, \ldots, x_7\}$ have the same units digit then their difference is divisible by 10. If this does not happen consider the ten possible units digits as follows: $\{0\}, \{1,9\}, \{2,8\}, \{3,7\}, \{4,6\}, \{5\}$ – these are the pigeonholes for the problem. When the seven pigeons $\{x_1, x_2, \ldots, x_7\}$ go to the pigeonholes where their units digits are located, at least one two-element subset is filled and those two numbers (pigeons) will sum to a multiple of 10.

15. For $\prod_{k=1}^{n}(k - i_k)$ to be odd, $(k - i_k)$ must be odd for all $1 \leq k \leq n$, i.e., one of k, i_k must be even and the other odd. Since n is odd, $n = 2m+1$ and in the list $1, 2, \ldots, n$, there are m even integers and $m+1$ odd integers. Let $1, 3, 5, \ldots, n$ be the pigeons and $i_1, i_3, i_5, \ldots, i_n$ the pigeonholes. At most m of the pigeonholes can be even integers, so $(k - i_k)$ must be even for at least one $k = 1, 3, 5, \ldots, n$. Consequently, $\prod_{k=1}^{n}(k - i_k)$ is even.

16. (a) The answer is the number of onto functions $f : A \longrightarrow B$ where $|A| = 10$ (weekly chores) and $|B| = 3$ (for the three young men). There are $3!S(10,3)$ such functions.

(b) $2!S(9,2)$ (Thomas only mows the lawn) $+3!S(9,3)$ (Thomas does more than just mow the lawn).

17. Let the n distinct objects be $x_1, x_2, \ldots, x_n$. Place x_n in a container. Now there are two *distinct* containers. For each of $x_1, x_2, \ldots, x_{n-1}$ there are two choices and this gives 2^{n-1} distributions. Among these there is one where $x_1, x_2, \ldots, x_{n-1}$ are in the container with x_n, so we remove this distribution and find $S(n,2) = 2^{n-1} - 1$.

18. (a) $\binom{12}{9}$ (b) $5!S(9,5)$

(c) $4!S(7,4)$ (Donald gets only the two books on basketball) $+\ 5!S(7,5)$ (Donald gets the two books on basketball and at least one other book.)

19. (a) and (b) $m!S(n,m)$

20. $S(n, n-2)$ is the number of ways to place n distinct objects into $n-2$ identical containers with no container left empty. There are two cases. One container contains three objects and the others one. This can happen in $\binom{n}{3}$ ways. The other possibility is that two containers each contain two objects and the others one. This happens in $(1/2)\binom{n}{2}\binom{n-2}{2} = (n!)/[2!2!2!(n-4)!] = 3\binom{n}{4}$ ways.

21. Fix $m = 1$. For $n = 1$ the result is true. Assume $f \circ f^k = f^k \circ f$ and consider $f \circ f^{k+1}$. $f \circ f^{k+1} = f \circ (f \circ f^k) = f \circ (f^k \circ f) = (f \circ f^k) \circ f = f^{k+1} \circ f$. Hence $f \circ f^n = f^n \circ f$ for all $n \in \mathbf{Z}^+$. Now assume that for $t \geq 1$, $f^t \circ f^n = f^n \circ f^t$. Then $f^{t+1} \circ f^n = (f \circ f^t) \circ f^n = f \circ (f^t \circ f^n) = f \circ (f^n \circ f^t) = (f \circ f^n) \circ f^t = (f^n \circ f) \circ f^t = f^n \circ (f \circ f^t) = f^n \circ f^{t+1}$, so $f^m \circ f^n = f^n \circ f^m$ for all $m, n \in \mathbf{Z}^+$.

22. (b) $y \in f(\bigcap_{i \in I} A_i) \iff y = f(x)$, for some $x \in \bigcap_{i\in I} A_i \implies$ $y \in f(A_i)$, for all $i \in I \iff y \in \bigcap_{i \in I} f(A_i)$.

(c) From part (b), $f(\bigcap_{i\in I} A_i) \subseteq \bigcap_{i \in I} f(A_i)$. For the opposite inclusion let $y \in \bigcap_{i\in I} f(A_i)$. Then $y \in f(A_i)$ for all $i \in I$, so $y = f(x_i), x_i \in A_i$, for each $i \in I$. Since f is one-to-one, all of these x_i's, $i \in I$, yield only one element $x \in \bigcap_{i\in I} A_i$. Hence $y = f(x) \in f(\bigcap_{i \in I} A_i)$, so $\bigcap_{i\in I} f(A_i) \subseteq f(\bigcap_{i\in I} A_i)$ and the equality follows.

The proof for part (a) is done in a similar way.

23. Proof: Let $a \in A$. Then

$$f(a) = g(f(f(a))) = f(g(f(f(f(a))))) = f(g \circ f^3(a)).$$

From $f(a) = g(f(f(a)))$ we have $f^2(a) = (f \circ f)(a) = f(g(f(f(a))))$. So $f(a) = f(g \circ f^3(a)) = f(g(f(f(f(a))))) = f^2(f(a)) = f^2(g(f^2(a))) = f(f(g(f(f(a))))) = f(g(f(a))) = g(a)$.

Consequently, $f = g$.

24. (a) $n^{(n \times n)} = n^{(n^2)}$ (b) $n^{(n^3)}$ (c) $n^{(n^k)}$

(d) Since $|A| = n$, there are n choices for each selection of size k, with repetitions allowed, from the set A of size n. There are $r = \binom{n+k-1}{k}$ possible selections and n^r commutative k-ary operations on A.

25. a) Note that $2 = 2^1$, $16 = 2^4$, $128 = 2^7$, $1024 = 2^{10}$, $8192 = 2^{13}$, and $65536 = 2^{16}$. Consider the exponents on 2. If four numbers are selected from $\{1, 4, 7, 10, 13, 16\}$, there is at least one pair whose sum is 17. Hence if four numbers are selected from S, there are two numbers whose product is $2^{17} = 131072$.

b) Let $a, b, c, d, n \in \mathbf{Z}^+$. Let $S = \{b^a, b^{a+d}, b^{a+2d}, \ldots, b^{a+nd}\}$. If $\lceil \frac{n}{2} \rceil + 1$ numbers are selected from S then there are at least two of them whose product is b^{2a+nd}.

26. (a) $\chi_{A\cap B}, \chi_A \cdot \chi_B$ both have domain $\mathcal{U}$ and codomain $\{0,1\}$. For each $x \in \mathcal{U}$, $\chi_{A \cap B}(x) = 1$ iff $x \in A \cap B$ iff $x \in A$ and $x \in B$ iff $\chi_A(x) = 1$ and $\chi_B(x) = 1$. Also, $\chi_{A\cap B}(x) =$

0 iff $x \notin A \cap B$ iff $x \notin A$ or $x \notin B$ iff $\chi_A(x) = 0$ or $\chi_B(x) = 0$ iff $\chi_A \cdot \chi_B(x) = 0$. Hence $\chi_{A \cap B} = \chi_A \cdot \chi_B$.

(b) The proof here is similar to that of part (a).

(c) $\chi_{\bar{A}}(x) = 1$ iff $x \in \bar{A}$ iff $x \notin A$ iff $\chi_A(x) = 0$ iff $(1 - \chi_A)(x) = 1$. $\chi_{\bar{A}}(x) = 0$ iff $x \notin \bar{A}$ iff $x \in A$ iff $\chi_A(x) = 1$ iff $(1 - \chi_A)(x) = 0$. Hence $\chi_{\bar{A}} = 1 - \chi_A$.

27. $f \circ g = \{(x,z),(y,y),(z,x)\}; g \circ f = \{(x,x),(y,z),(z,y)\};$
$f^{-1} = \{(x,z),(y,x),(z,y)\}; g^{-1} = \{(x,y),(y,x),(z,z)\};$
$(g \circ f)^{-1} = \{(x,x),(y,z),(z,y)\} = f^{-1} \circ g^{-1}; g^{-1} \circ f^{-1} = \{(x,z),(y,y),(z,x)\}.$

28. (a) $f^{-1}(8) = \{x|5x + 3 = 8\} = \{1\}$.

(b) $|x^2 + 3x + 1| = 1 \Longrightarrow x^2 + 3x + 1 = 1$ or $x^2 + 3x + 1 = -1 \Longrightarrow x^2 + 3x = 0$ or $x^2 + 3x + 2 = 0 \Longrightarrow (x)(x+3) = 0$ or $(x+2)(x+1) = 0 \Longrightarrow x = 0, -3$ or $x = -1, -2$. Hence $g^{-1}(1) = \{-3, -2, -1, 0\}$.

(c) $\{-8/5, -8/3\}$

29. Under these conditions we know that $f^{-1}(\{6,7,9\}) = \{2,4,5,6,9\}$. Consequently we have
(i) two choices for each of $f(1)$, $f(3)$, and $f(7)$ – namely, 4 or 5;
(ii) two choices for each of $f(8)$ and $f(10)$ – namely, 8 or 10; and
(iii) three choices for each of $f(2)$, $f(4)$, $f(5)$, $f(6)$, and $f(9)$ – namely, 6, 7, or 9.
Therefore, by the rule of product, it follows that the number of functions satisfying these conditions is $2^3 \cdot 2^2 \cdot 3^5 = 7776$.

30. Since $f^1 = f$ and $(f^{-1})^1 = f^{-1}$, the result is true for $n = 1$. Assume the result for $n = k : (f^k)^{-1} = (f^{-1})^k$. For $n = k+1, (f^{k+1})^{-1} = (f \circ f^k)^{-1} = (f^k)^{-1} \circ (f^{-1}) = (f^{-1})^k \circ (f^{-1})^1 = (f^{-1})^1 \circ (f^{-1})^k$ (by Exercise 21) $= (f^{-1})^{k+1}$. Therefore, by the Principle of Mathematical Induction, the result is true for all $n \in \mathbf{Z}^+$.

31. (a) $(\pi \circ \sigma)(x) = (\sigma \circ \pi)(x) = x$

(b) $\pi^n(x) = x - n; \sigma^n(x) = x + n (n \geq 2)$.

(c) $\pi^{-n}(x) = x + n; \sigma^{-n}(x) = x - n (n \geq 2)$.

32. (a) $\tau(n) = (e_1 + 1)(e_2 + 1) \cdots (e_k + 1)$

(b) $k = 2 : \tau(2) = \tau(3) = \tau(5) = 2$
$k = 3 : \tau(2^2) = \tau(3^2) = \tau(5^2) = 3$
$k = 4 : \tau(6) = \tau(8) = \tau(10) = 4$
$k = 5 : \tau(2^4) = \tau(3^4) = \tau(5^4) = 5$
$k = 6 : \tau(12) = \tau(18) = \tau(20) = 6$

(c) For all $k > 1$ and any prime p, $\tau(p^{k-1}) = k$.

(d) Let $a = p_1^{e_1} p_2^{e_2} \cdots p_k^{e_k}$ and $b = q_1^{f_1} q_2^{f_2} \cdots q_t^{f_t}$, where $p_1, p_2, \ldots, p_k, q_1, q_2, \ldots, q_t$ are $k + t$ distinct primes, and $e_1, e_2, \ldots, e_k, f_1, f_2, \ldots f_t \in \mathbf{Z}^+$. Then
$\tau(ab) = (e_1 + 1)(e_2 + 1) \cdots (e_k + 1)(f_1 + 1)(f_2 + 1) \cdots (f_t + 1)$

$= [(e_1+1)(e_2+1)\cdots(e_k+1)][(f_1+1)(f_2+1)\cdots(f_t+1)] = \tau(a)\tau(b).$

33. (a) Here there are eight distinct primes and each subset A satisfying the stated property determines a distribution of the eight distinct objects in $X = \{2,3,5,7,11,13,17,19\}$ into four identical containers with no container left empty. There are $S(8,4)$ such distributions.

(b) $S(n,m)$

34. Define $f: \mathbf{Z}^+ \to \mathbf{R}$ by $f(n) = 1/n$.

35. (a) Let $m = 1$ and $k = 1$. Then for all $n \geq k, |f(n)| \leq 2 < 3 \leq |g(n)| = m|g(n)|$, so $f \in O(g)$.

(b) Let $m = 4$ and $k = 1$. Then for all $n \geq k, |g(n)| \leq 4 = 4 \cdot 1 \leq 4|f(n)| = m|f(n)|$, so $g \in O(f)$.

36. (a)$f \in O(f_1) \Longrightarrow \exists m_1 \in \mathbf{R}^+ \; \exists k_1 \in \mathbf{Z}^+$ such that $|f(n)| \leq m_1|f_1(n)| \; \forall n \geq k_1$.
$g \in O(g_1) \Longrightarrow \exists m_2 \in \mathbf{R}^+ \; \exists k_2 \in \mathbf{Z}^+$ such that $|g(n)| \leq m_2|g_1(n)| \; \forall n \geq k_2$.
Let $m = max\{m_1, m_2\}$. Then for all $n \geq max\{k_1, k_2\}, |(f+g)(n)| = |f(n)+g(n)| = |f(n)| + |g(n)| \leq m_1|f_1(n)| + m_2|g_1(n)| \leq m(|f_1(n)| + |g_1(n)|) = m|f_1(n) + g_1(n)| = m|(f_1+g_1)(n)|$, so $(f+g) \in O(f_1+g_1)$.

(b) Let $f, f_1, g, g_1 : \mathbf{Z}^+ \longrightarrow \mathbf{R}$ be defined by $f(n) = n, f_1(n) = 1-n, g(n) = 1, g_1(n) = n$.

37. First note that if $\log_a n = r$, then $n = a^r$ and $\log_b n = \log_b(a^r) = r\log_b a = (\log_b a)(\log_a n)$. Now let $m = (\log_b a)$ and $k = 1$. Then for all $n \geq k, |g(n)| = \log_b n = (\log_b a)(\log_a n) = m|f(n)|$, so $g \in O(f)$.
Finally, with $m = (\log_b a)^{-1} = \log_a b$ and $k = 1$, we find that for all $n \geq k, |f(n)| = \log_a n = (\log_a b)(\log_b n) = m|g(n)|$. Hence $f \in O(g)$.

CHAPTER 6
LANGUAGES: FINITE STATE MACHINES

Section 6.1

1. (a) 25; 125 (b) $\sum_{i=0}^{5} 5^i = 3906$

2. (a) 4^4 (b) $\binom{5}{2}(3^3)$ (c) 3^5 (d) $3^5 + \binom{5}{2}3^3 + \binom{5}{4}3$

3. 12

4.

(a) 0	(b) 0	(c) 1	(d) 2
(e) 3	(f) 4	(g) 1	(h) 0

5. $\sum_{i=1}^{4} 5^i$

6. There are 100 substrings of length 1: $x_1, x_2, \ldots, x_{100}$; 99 substrings of length 2: $x_1x_2, x_2x_3, \ldots, x_{99}x_{100}$; ... ; 1 substring of length 100: $x_1x_2 \ldots x_{100}$. So there are $100 + 99 + \ldots + 1 = \sum_{i=1}^{100} i = (100)(101)/2 = 5050$ nonempty substrings in total.

7. (a) { 00,11, 000,111, 0000,1111}
(b) {0,1}
(c) $\sum^* - \{\lambda, 00, 11, 000, 111, 0000, 1111\}$
(d) {0,1, 00,11}
(e) $\sum^*$
(f) $\sum^* - \{0, 1, 00, 11\} = \{\lambda, 01, 10\} \cup \{w | \parallel w \parallel \geq 3\}$

8. (a) $AB = \{1000, 101, 1100, 111\}$
(b) $BA = \{0010, 0011, 110, 111\}$
(c) $A^3 = \{101010, 101011, 101110, 111010, 101111, 111011, 111110, 111111\}$
(d) $B^2 = \{0000, 001, 100, 11\}$.

9. (a) $x \in AC \Longrightarrow x = ac$, for some $a \in A, c \in C \Longrightarrow x \in BD$, since $A \subseteq B, C \subseteq D$.

(b) If $A\emptyset \neq \emptyset$, let $x \in A\emptyset$. $x \in A\emptyset \Longrightarrow x = yz$, for some $y \in A, z \in \emptyset$. But $z \in \emptyset$ is impossible. Hence $A\emptyset = \emptyset$. [In like manner $\emptyset A = \emptyset$].

10. (a) $AB = \{xy, xyx\}$ (b) $BA = \{xy, xxy\}$ (c) $B^3 = \{\lambda, x, x^2, x^3\}$
(d) $B^+ = \{x^n | n \in \mathbf{Z}^+\}$ (e) $A^* = \{\lambda, xy, xyxy, \ldots\} = \{\lambda\} \cup \{(xy)^n | n \in \mathbf{Z}^+\}$

11. For any alphabet $\sum$, let $B \subseteq \sum$. Then, if $A = B^*$, it follows from part (f) of Theorem 6.2 that $A^* = (B^*)^* = B^* = A$.

12.

(a) Yes (b) Yes (c) Yes
(d) Yes (e) No (f) Yes

13. (a) Here A^* consists of all strings x of even length where if $x \neq \lambda$, then x starts with 0 and ends with 1, and the symbols (0 and 1) alternate.
(b) In this case A^* contains precisely those strings made up of $3n$ 0's, for $n \in \mathbf{N}$.
(c) Here a string $x \in A^*$ if (and only if)
(i) x is a string of n 0's, for $n \in \mathbf{N}$; or
(ii) x is a string that starts and ends with 0, and has at least one 1 – but no consecutive 1's.
(d) For this last case A^* consists of the following:
(i) Any string of n 1's, for $n \in \mathbf{N}$; and
(ii) Any string that starts with 1 and contains at least one 0, but no consecutive 0's.

14. There are five possible choices:
(1) $A = \{\lambda\}$, $B = \{01, 000, 0101, 0111, 01000, 010111\}$;
(2) $A = \{01, 000, 0101, 0111, 01000, 010111\}$, $B = \{\lambda\}$;
(3) $A = \{0\}$, $B = \{1, 00, 101, 111, 1000, 10111\}$;
(4) $A = \{0, 010\}$, $B = \{1, 00, 111\}$; and
(5) $A = \{\lambda, 01\}$, $B = \{01, 000, 0111\}$.

15. Let $\sum$ be an alphabet with $\emptyset \neq A \subseteq \sum^*$. If $|A| = 1$ and $x \in A$, then $xx = x$ since $A^2 = A$. But $\| xx \| = 2 \| x \| = \| x \| \Longrightarrow \| x \| = 0 \Longrightarrow x = \lambda$. If $|A| > 1$, let $x \in A$ where $\| x \| > 0$ but $\| x \|$ is minimal. Then $x \in A^2 \Longrightarrow x = yz$, for some $y, z \in A$. Since $\| x \| = \| y \| + \| z \|$, if $\| y \|, \| z \| > 0$, then one of y, z is in A with length smaller than $\| x \|$. Consequently, one of $\| y \|$ or $\| z \|$ is 0, so $\lambda \in A$.

16. (a) p_a, s_a, r
(b) r, d are onto; $p_a(\sum^*) = \{a\}\sum^*$; $s_a(\sum^*) = \sum^*\{a\}$
(c) r is invertible and $r^{-1} = r$.
(d) $25; 125; 5^{n/2}$ for n even, $5^{(n+1)/2}$ for n odd.
(e) $(d \circ p_a)(x) = x = (r \circ d \circ r \circ s_a)(x)$
(f) $r^{-1}(B) = \{ea, ia, oa, oo, oie, uuoie\}$
$p_a^{-1}(B) = \{e, i, o\}$.
$s_a^{-1}(B) = \emptyset$
$|d^{-1}(B)| = |\bigcup_{x \in B} d^{-1}(x)| = \sum_{x \in B} d^{-1}(x) = \sum_{x \in B} 5 = 6(5) = 30$

17. If $A = A^2$ then it follows by mathematical induction that $A = A^n$ for all $n \in \mathbf{Z}^+$. Hence $A = A^+$. From Exercise 15 we know that $A = A^2 \Longrightarrow \lambda \in A$, so $A = A^*$.

18. Theorem 6.1(b): $x \in (AB)C \Longleftrightarrow x = (ab)c$, for some $a \in A, b \in B, c \in C \Longleftrightarrow$ $x = (a_1 a_2 \ldots a_\ell b_1 b_2 \ldots b_m)(c_1 c_1 \ldots c_n)$, where $a_i \in A, 1 \leq i \leq \ell$, $b_j \in B, 1 \leq j \leq m$, $c_k \in C, 1 \leq k \leq n \Longleftrightarrow x = a_1 a_2 \ldots a_\ell b_1 b_2 \ldots b_m c_1 c_2 \ldots c_n$, where $a_i \in A, 1 \leq i \leq \ell$, $b_j \in B, 1 \leq j \leq m$, $c_k \in C, 1 \leq k \leq n \Longleftrightarrow x = (a_1 a_2 \ldots a_\ell)(b_1 b_1 \ldots b_m c_1 c_2 \ldots c_n)$, where

$a_i \in A, 1 \leq i \leq \ell,\ b_j \in B, 1 \leq j \leq m,\ c_k \in C, 1 \leq k \leq n \Longleftrightarrow x \in A(BC)$. Hence $(AB)C = A(BC)$.

Theorem 6.2(b): For $a \in A, a = \lambda a$ with $\lambda \in B^*$. Hence $A \subseteq B^*A$.

Theorem 6.2(f): From Theorem 6.2(a) $A^* \subseteq A^*A^*$. Conversely, $x \in A^*A^* \Longrightarrow x = yz$ where $y = a_1a_2 \ldots a_m, z = a_1'a_2' \ldots a_n'$, with $a_i, a_j' \in A$, for $1 \leq i \leq m, 1 \leq j \leq n$. Hence $x \in A^*$, so $A^*A^* \subseteq A^*$ and the equality follows.

Since $(A^*)^* = \bigcup_{n=0}^{\infty}(A^*)^n$, it follows that $A^* \subseteq (A^*)^*$. Conversely, if $x \in (A^*)^*$, then $x = x_1x_2 \ldots x_n$, where $x_i \in A^*$, for $1 \leq i \leq n$. Each $x_i = a_{i1}a_{i2} \ldots a_{ik_i}$, where $a_{ij} \in A, 1 \leq j \leq k_i$. Hence $x \in A^*$, so $(A^*)^* \subseteq A^*$ and $(A^*)^* = A^*$.

$(A^*)^+ = \bigcup_{n=1}^{\infty}(A^*)^n \subseteq \bigcup_{n=0}^{\infty}(A^*)^n = (A^*)^*$. If $x \in (A^*)^*$, then $x = x_1x_2 \ldots x_n$, where $x_i \in A^*$, for $1 \leq i \leq n$. Then $x = a_{11}a_{12} \ldots a_{1k_1}a_{21}a_{22} \ldots a_{2k_2} \ldots a_{n1} \ldots a_{nk_n} \in A^* \subseteq \bigcup_{n=1}^{\infty}(A^*)^n = (A^*)^+$, so $(A^*)^* = (A^*)^+$.

Since $A^+ \subseteq A^*, (A^+)^* \subseteq (A^*)^*$ by part (d) of this theorem. For $x \in (A^*)^*$, if $x = \lambda$, then $x \in (A^+)^*$. If $x \neq \lambda$, then as above $x = a_{11}a_{12} \ldots a_{1k_1}a_{21} \ldots a_{2k_2} \ldots a_{n1} \ldots a_{nk_n} \in A^+ \subseteq (A^+)^*$ and the result follows.

19. By Definition 6.11 $AB = \{ab | a \in A, b \in B\}$, and since it is possible to have $a_1b_1 = a_2b_2$ with $a_1, a_2 \in A, a_1 \neq a_2$, and $b_1, b_2 \in B, b_1 \neq b_2$, it follows that $|AB| \leq |A \times B| = |A||B|$.

20. (a) $\{y\}^*x\{y\}^*$ (b) $\{y\}^*x\{y\}^*x\{y\}^*$ (c) $x\{x, y\}^*$

(d) $\{x, y\}^*yxy$ (e) $(x\{x, y\}^*) \cup (\{x, y\}^*yxy)$

(f) $[(x\{x, y\}^*) \cup (\{x, y\}^*yxy)] - [x\{x, y\}^*yxy]$

21. (a) The words 001 and 011 have length 3 and are in A. The words 00011 and 00111 have length 5 and they are also in A.

(b) From step (1) we know that $1 \in A$. Then by applying step (2) three times we get

(i) $1 \in A \Longrightarrow 011 \in A$;

(ii) $011 \in A \Longrightarrow 00111 \in A$; and

(iii) $00111 \in A \Longrightarrow 0001111 \in A$.

(c) If 00001111 were in A, then from step (2) we see that this word would have to be generated from 000111 (in A). Likewise, 000111 in $A \Longrightarrow 0011$ is in $A \Longrightarrow 01$ is in A. However, there are no words in A of length 2 — in fact, there are no words of even length in A.

22. (a) (1) $\lambda \in A$; and

(2) If $x \in A$, then each of the following is also in A:

(i) $x1$ (ii) $1x$ (iii) $00x$ (iv) $x00$ (v) $0x0$

[And no other string of 0's and 1's is in A.]

(b) (1) $\lambda \in A$; and

(2) For each $x \in A$ the strings $1x$ and $x0$ are also in A.

23.

(a) **Steps**	**Reasons**
1. ( ) is in A.	Part (1) of the recursive definition
2. (( )) is in A.	Step 1 and part (2(ii)) of the definition
3. (( ))( ) is in A.	Steps 1, 2, and part (2(i)) of the definition

(b) **Steps**	**Reasons**
1. ( ) is in A.	Part (1) of the recursive definition
2. (( )) is in A.	Step 1 and part (2(ii)) of the definition
3. (( ))( ) is in A.	Steps 1, 2, and part (2(i)) of the definition
4. (( ))( )( ) is in A.	Steps 1, 3, and part (2(i)) of the definition

(c) **Steps**	**Reasons**
1. ( ) is in A.	Part (1) of the recursive definition
2. ( )( ) is in A.	Step 1 and part (2(i)) of the definition
3. (( )( )) is in A.	Step 2 and part (2(ii)) of the definition
4. ( )(( )( )) is in A.	Steps 1, 3, and part (2(i)) of the definition

24. (1) $\lambda \in A$ and $s \in A$ for all $s \in \Sigma$; and
(2) For each $x \in A$ and $s \in \Sigma$, the string sxs is also in A.
[No other string from Σ^* is in A.]

25. Length 3: $\binom{3}{0} + \binom{2}{1} = 3$
Length 4: $\binom{4}{0} + \binom{3}{1} + \binom{2}{2} = 5$
Length 5: $\binom{5}{0} + \binom{4}{1} + \binom{3}{2} = 8$
Length 6: $\binom{6}{0} + \binom{5}{1} + \binom{4}{2} + \binom{3}{3} = 13$ [Here the summand $\binom{6}{0}$ counts the strings where there are no 0s; the summand $\binom{5}{1}$ counts the strings where we arrange the symbols $1, 1, 1, 1, 00$; the summand $\binom{4}{2}$ is for the arrangements of $1, 1, 00, 00$; and the summand $\binom{3}{3}$ counts the arrangements of $00, 00, 00$.]

26. [Here $\binom{9}{1}$ counts the arrangements for one 111 and eight 00's; $\binom{8}{3}$ counts the arrangements for three 111's and five 00's; and $\binom{7}{5}$ is for the arrangements of five 111's and two 00's.]

27.

A: (1) $\lambda \in A$
(2) If $a \in A$, then $0a0, 0a1, 1a0, 1a1 \in A$.

B: (1) $0, 1 \in A$.
(2) If $a \in A$, then $0a0, 0a1, 1a0, 1a1 \in A$.

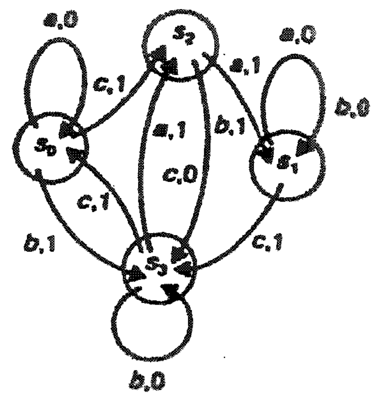
28. Of the $3 \cdot 3 \cdot 3 \cdot 3 = 3^4 = 81$ words in $\sum^4$, there are $3 \cdot 3 \cdot 3 \cdot 2 = 27 \cdot 2 = 54$ words that start with one of the letters a, b, or c and end with a different letter. Consequently, one must select at least $54 + 2 = 56$ words from $\sum^4$ to guarantee that at least two start and end with the same letter.

Section 6.2

1. (a) 0010101; s_1 (b) 0000000; s_1 (c) 001000000; s_0

2. Because of the first output of 1 we must be at state s_2 when the third input is read. This then forces the first three inputs to be 1,0,1. To get the second output of 1 we must be at state s_2 when the fifth input is read. This forces the remaining two inputs to be 0,1. Hence $x = 10101$.

3. (a) 010110 (b)



4. $S = \{s_i | 0 \leq i \leq 5\}$, where at state s_i, the machine remembers the insertion of a total of $5i$ cents.

$I = \{5¢, 10¢ , 25¢, B,W\}$

$O = \{$n (nothing), P(peppermint), S(spearmint), 5¢, 10¢, 15¢, 20¢, 25¢$\}$

	ν					ω				
	5¢	10¢	25¢	B	W	5¢	10¢	25¢	B	W
s_0	s_1	s_2	s_5	s_0	s_0	n	n	n	n	n
s_1	s_2	s_3	s_5	s_1	s_1	n	n	5¢	n	n
s_2	s_3	s_4	s_5	s_2	s_2	n	n	10¢	n	n
s_3	s_4	s_5	s_5	s_3	s_3	n	n	15¢	n	n
s_4	s_5	s_5	s_5	s_4	s_4	n	5¢	20¢	n	n
s_5	s_5	s_5	s_5	s_0	s_0	5¢	10¢	25¢	S	P

5. (a) $010000; s_2$ (b) $(s_1)100000; s_2$ $(s_2)000000; s_2$ $(s_3)110010; s_2$

(c)

	ν		ω	
	0	1	0	1
s_0	s_0	s_1	0	0
s_1	s_1	s_2	1	1
s_2	s_2	s_2	0	0
s_3	s_0	s_3	0	1
s_4	s_2	s_3	0	1

(d) s_1 (e) $x = 101$ (unique)

6. (a) The machine recognizes (with an output of 1) every 0 (in an input string x) that is preceded by another 0.

(b) State s_1 remembers that at least one 0 has been supplied from an input string x.

(c) $A = \{1\}^*, B = \{00\}$

7. (a) (i) 15 (ii) 3^{15} (iii) 2^{15} (b) 6^{15}

8. (a)

Input	0	1	1	0	1	1	1	0	1	1
Output	0	0	0	0	0	0	0	0	1	0

(b)

	ν		ω	
	0	1	0	1
s_0	s_0	s_1	0	0
s_1	s_1	s_2	0	0
s_2	s_2	s_3	0	0
s_3	s_3	s_4	0	0
s_4	s_4	s_5	0	0
s_5	s_5	s_6	0	1

(c) $\omega(x, s_0) = 0000001$ for $x = (1)1111101; (2)1111011; (3)1110111; (4)1101111;$ $(5)1011111;$ and $(6)0111111$

(d) The machine recognizes the occurrence of a sixth 1, a 12th 1, ... in an input x.

9.

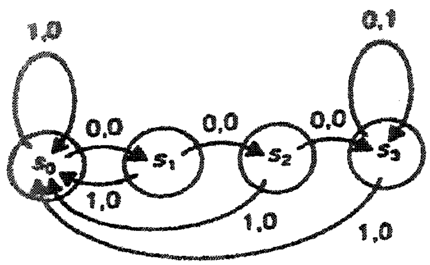
(a)

	ν		ω	
	0	1	0	1
s_0	s_4	s_1	0	0
s_1	s_3	s_2	0	0
s_2	s_3	s_2	0	1
s_3	s_3	s_3	0	0
s_4	s_5	s_3	0	0
s_5	s_5	s_3	1	0

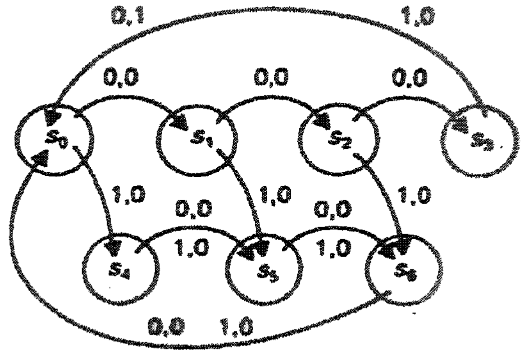
(b) There are only two possibilities: $x = 1111$ or $x = 0000$.
(c) $A = \{111\}\{1\}^* \cup \{000\}\{0\}^*$
(d) Here $A = \{11111\}\{1\}^* \cup \{00000\}\{0\}^*$.

Section 6.3

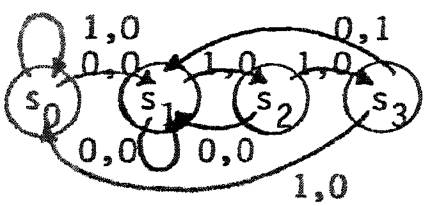
1. (a)

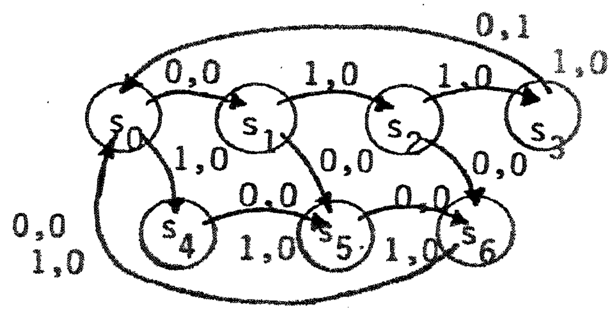


(b)

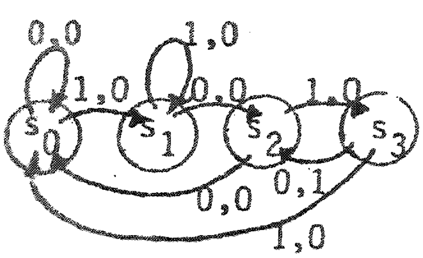


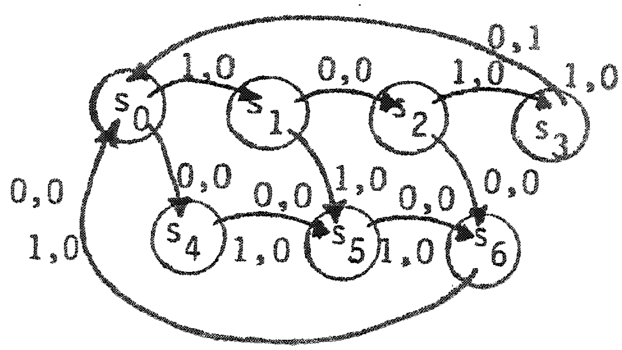
2. (0110)



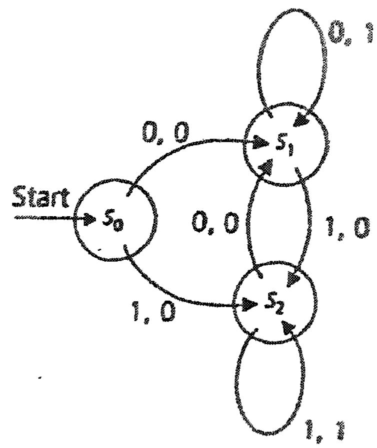


(1010)

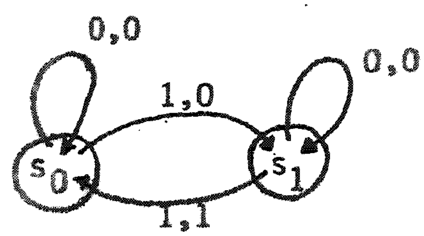




3.

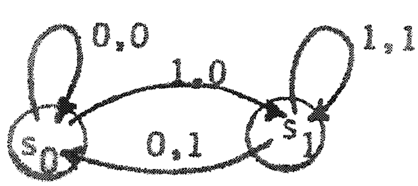


4.



5.

(a)



(b) (i) Input 111
Output 011
(ii) Input 1010
Output 0101
(iii) Input 00011
Output 00001

(c) The machine outputs a 0 followed by the first $n-1$ symbols of the n symbol input string x. Hence the machine is a unit delay.

(d) The machine here performs the same tasks as the one in Fig. 6.13 (and has only two states.)

6. Suppose the contrary and let the machine have n states, for some $n \in \mathbf{Z}^+$. Consider the input string $0^{n+1}1^n$. We expect the output here to be $0^{n+1}1^n$. As the 0's in this input string are processed we obtain $n+1$ states $s_1, s_2, \ldots, s_n, s_{n+1}$ from the function ν. Consequently, by the Pigeonhole Principle, there are two states s_i, s_j where $i < j$ but $s_i = s_j$. So if the states s_m, for $i+1 \leq m \leq j$, are removed, along with their inputs of 0, then this machine will recognize the sequence $0^{n+1-(j-i)}1^n$, where $n+1-(j-i) \leq n$. But the string $0^{n+1-(j-i)}1^n \notin A$.

7. (a) The transient states are s_0, s_1. State s_4 is a sink state. $\{s_1, s_2, s_3, s_4, s_5\}$, $\{s_4\}$, $\{s_2, s_3, s_5\}$ (with the corresponding restrictions on the given function ν) constitute submachines. The strongly connected submachines are $\{s_4\}$ and $\{s_2, s_3, s_5\}$.

(b) States s_2, s_3 are transient. The only sink state is s_4. The set $\{s_0, s_1, s_3, s_4\}$ provides the states for a submachine; $\{s_0, s_1\}$, $\{s_4\}$ provide strongly connected submachines.

(c) Here there are no transient states. State s_6 is a sink state. There are three submachines: $\{s_2, s_3, s_4, s_5, s_6\}$, $\{s_3, s_4, s_5, s_6\}$, and $\{s_6\}$. The only strongly connected submachine is $\{s_6\}$.

8. Either 110 or 111 provides a transfer sequence from s_2 to s_5.

Supplementary Exercises

1. (a) True (b) False (c) True
(d) True (e) True (f) True

2. No. Let $x \in \sum$ with $A = \{x, xx\}, B = \{x\}$. Then $A^* = B^* = \{x^n | n \geq 0\}$, but $A \not\subseteq B$.

3. Let $x \in \sum$ and $A = \{x\}$. Then $A^2 = \{x^2\}$ and $(A^2)^* = \{\lambda, x^2, x^4, \ldots\}$. However $A^* = (\lambda, x, x^2, \ldots\}$ and $(A^*)^2 = A^*$, so $(A^*)^2 \neq (A^2)^*$.

4. (a) $A^* \subset B^*$. [For example, $111 \in B^*$ but $111 \notin A^*$.]

(b) $A^* = C^*$.

5. O_{02} : Starting at s_0 we can return to s_0 for any input from $\{1, 00\}^*$. To finish at state s_2 requires an input of 0. Hence $O_{02} = \{1, 00\}^*\{0\}$
$O_{22} : \{0\}\{1, 00\}^*\{0\}$
$O_{11} : \emptyset$
$O_{00} : \{1, 00\}^* - \{\lambda\}$
$O_{10} : \{1\}\{1, 00\}^* \cup \{10\}\{1, 00\}^*$

6. (a)

	ν		ω	
	0	1	0	1
s_0	s_0	s_1	0	0
s_1	s_1	s_2	0	0
s_2	s_2	s_3	0	0
s_3	s_3	s_0	0	1

(b) For any input string x, this machine recognizes (with output 1) the occurrence of every fourth 1 in x.

(c) $\binom{8}{8} + \binom{8}{4} + \binom{8}{0} = 72$. (The first summand is for the sequence of eight 1's, the second summand for the sequences of four 1's and four 0's, and the last summand for the sequence of eight 0's.)

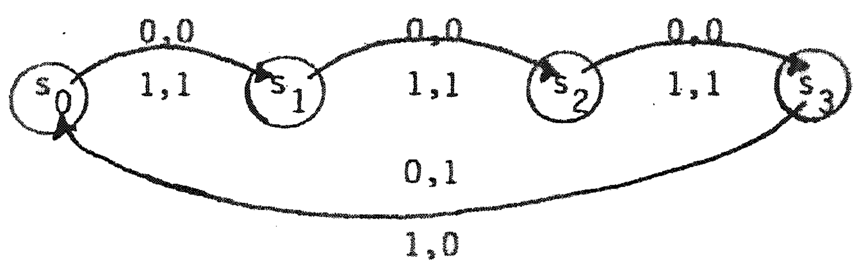
For $\| x \| = 12$, there are $\binom{12}{12} + \binom{12}{8} + \binom{12}{4} + \binom{12}{0} = 992$ such sequences.

7. (a) By the Pigeonhole Principle there is a first state s that is encountered twice. Let y be the output string that resulted since s was first encountered until we reach this state a second time. Then from that point on the output is $yyy\ldots$.

(b) n (c) n

8. $x = 110$

9.



10.

	ν		ω	
	0	1	0	1
s_0	s_1	s_2	0	1
s_1	s_2	s_1	1	0
s_2	s_2	s_2	1	0
s_3	s_1	s_0	0	1

Here the table for ω is obtained from Table 6.15 by reversing 0 and 1 (and, 1 and 0) for the columns under 0 and 1.

11.

(a)

	ν		ω	
	0	1	0	1
(s_0, s_3)	(s_0, s_4)	(s_1, s_3)	1	1
(s_0, s_4)	(s_0, s_3)	(s_1, s_4)	0	1
(s_1, s_3)	(s_1, s_3)	(s_2, s_4)	1	1
(s_1, s_4)	(s_1, s_4)	(s_2, s_3)	1	1
(s_2, s_3)	(s_2, s_3)	(s_0, s_4)	1	1
(s_2, s_4)	(s_2, s_4)	(s_0, s_3)	1	0

(b) $\omega((s_0, s_3), 1101) = 1111$; M_1 is in state s_0 and M_2 is in state s_4.

12. The following program determines the output for the input string 1000011000.

```
10    Dim A(3,2), B(3,2)
20    Mat Read A,B
30    Data 2,1,3,1,3,1,0,0,0,0,1,1
40    Dim P(100), S(100)
50    Read N
60    For I = 1 to N
70          Read X
80          If I <> 1 Then 120
90          If X = 0 Then P(1) = B(1,1) Else P(1) = B(1,2)
100         If X = 0 Then S(1) = A(1,1) Else S(1) = A(1,2)
110         Go To 140
120         Y = X + 1
130         P(I) = B(S(I-1)Y) : S(I) = A(S(I-1),Y)
140   Next I
150   Data 10,1,0,0,0,0,1,1,0,0,0
160   Print "The output is";
170   For I = 1 To N-1
180         Print P(I);
190   Next I
200   Print P(N)
210   Print "The machine is now in state"; S(N)
220   End
```

CHAPTER 7
RELATIONS: THE SECOND TIME AROUND

Section 7.1

1. (a) {(1,1),(2,2),(3,3),(4,4),(1,2),(2,1),(2,3),(3,2)}
   (b) {(1,1),(2,2),(3,3),(4,4),(1,2)}
   (c) {(1,1),(2,2),(1,2),(2,1)}

2. $-9, -2, 5, 12, 19$

3. (a) Let $f_1, f_2, f_3 \in F$ with $f_1(n) = n + 1$, $f_2(n) = 5n$, and $f_3(n) = 4n + 1/n$.
   (b) Let $g_1, g_2, g_3 \in F$ with $g_1(n) = 3$, $g_2(n) = 1/n$, and $f_3(n) = \sin n$.

4. (a) The relation $\mathcal{R}$ on the set A is
   (i) reflexive if $\forall x \in A\ (x, x) \in \mathcal{R}$
   (ii) symmetric if $\forall x, y \in A\ [(x, y) \in \mathcal{R} \Longrightarrow (y, x) \in \mathcal{R}]$
   (iii) transitive if $\forall x, y, z \in A\ [(x, y), (y, z) \in \mathcal{R} \Longrightarrow (x, z) \in \mathcal{R}]$
   (iv) antisymmetric if $\forall x, y \in A\ [(x, y), (y, x) \in \mathcal{R} \Longrightarrow x = y]$.

   (b) The relation $\mathcal{R}$ on the set A is
   (i) not reflexive if $\exists x \in A\ (x, x) \notin \mathcal{R}$
   (ii) not symmetric if $\exists x, y \in A\ [(x, y) \in \mathcal{R} \wedge (y, x) \notin \mathcal{R}]$
   (iii) not transitive if $\exists x, y, z \in A\ [(x, y), (y, z) \in \mathcal{R} \wedge (x, z) \notin \mathcal{R}]$
   (iv) not antisymmetric if $\exists x, y \in A\ [(x, y), (y, x) \in \mathcal{R} \wedge x \neq y]$.

5. (a) reflexive, antisymmetric, transitive
   (b) transitive
   (c) reflexive, symmetric, transitive
   (d) symmetric
   (e) (odd): symmetric
   (f) (even): reflexive, symmetric, transitive
   (g) reflexive, symmetric
   (h) reflexive, transitive

6. The relation in part (a) is a partial order. The relations in parts (c) and (f) are equivalence relations.

7. (a) For all $x \in A, (x, x) \in \mathcal{R}_1, \mathcal{R}_2$, so $(x, x) \in \mathcal{R}_1 \cap \mathcal{R}_2$ and $\mathcal{R}_1 \cap \mathcal{R}_2$ is reflexive.

(b) All of these results are true. For example if $\mathcal{R}_1, \mathcal{R}_2$ are both transitive and $(x,y),(y,z) \in \mathcal{R}_1 \cap \mathcal{R}_2$ then $(x,y),(y,z) \in \mathcal{R}_1, \mathcal{R}_2$, so $(x,z) \in \mathcal{R}_1, \mathcal{R}_2$ (transitive property) and $(x,z) \in \mathcal{R}_1 \cap \mathcal{R}_2$. [The proofs for the symmetric and antisymmetric properties are similar.]

8. (a) For all $x \in A, (x,x) \in \mathcal{R}_1, \mathcal{R}_2 \subseteq \mathcal{R}_1 \cup \mathcal{R}_2$, so if either $\mathcal{R}_1$ or $\mathcal{R}_2$ is reflexive, then $\mathcal{R}_1 \cup \mathcal{R}_2$ is reflexive.

(b) (i) If $x, y \in A$ and $(x,y) \in \mathcal{R}_1 \cup \mathcal{R}_2$, assume without loss of generality, that $(x,y) \in \mathcal{R}_1$. $(x,y) \in \mathcal{R}_1$ and $\mathcal{R}_1$ symmetric $\Longrightarrow (y,x) \in \mathcal{R}_1 \Longrightarrow (y,x) \in \mathcal{R}_1 \cup \mathcal{R}_2$, so $\mathcal{R}_1 \cup \mathcal{R}_2$ is symmetric.

(ii) False: Let $A = \{1,2\}, \mathcal{R}_1 = \{(1,1),(1,2)\}, \mathcal{R}_2 = \{(2,1)\}$. Then $(1,2),(2,1) \in \mathcal{R}_1 \cup \mathcal{R}_2$, and $1 \neq 2$, so $\mathcal{R}_1 \cup \mathcal{R}_2$ is not antisymmetric.

(iii) False: Let $A = \{1,2,3\}, \mathcal{R}_1 = \{(1,1),(1,2)\}, \mathcal{R}_2 = \{(2,3)\}$. Then $(1,2),(2,3) \in \mathcal{R}_1 \cup \mathcal{R}_2$, but $(1,3) \notin \mathcal{R}_1 \cup \mathcal{R}_2$, so $\mathcal{R}_1 \cup \mathcal{R}_2$ is not transitive.

9.

(a) False: Let $A = \{1,2\}$ and $\mathcal{R} = \{(1,2),(2,1)\}$.

(b)
(i) Reflexive: True
(ii) Symmetric: False. Let $A = \{1,2\}, \mathcal{R}_1 = \{(1,1)\}, \mathcal{R}_2 = \{(1,1),(1,2)\}$.
(iii) Antisymmetric & Transitive: False. Let $A = \{1,2\}, \mathcal{R}_1 = \{(1,2)\}$, $\mathcal{R}_2 = \{(1,2),(2,1)\}$.

(c)
(i) Reflexive: False. Let $A = \{1,2\}, \mathcal{R}_1 = \{(1,1)\}, \mathcal{R}_2 = \{(1,1),(2,2)\}$.
(ii) Symmetric: False. Let $A = \{1,2\}, \mathcal{R}_1 = \{(1,2)\}, \mathcal{R}_2 = \{(1,2),(2,1)\}$.
(iii) Antisymmetric: True
(iv) Transitive: False. Let $A = \{1,2\}, \mathcal{R}_1 = \{(1,2),(2,1)\}, \mathcal{R}_2 = \{(1,1),(1,2),(2,1),(2,2)\}$.

(d) True

10.

(a) 2^{12}	(b) $(2^4)(2^6) = 2^{10}$	(c) 2^6
(d) 2^{11}	(e) $(2^4)(2^5) = 2^9$	(f) $2^4 \cdot 3^6$
(g) $2^4 \cdot 3^5$	(h) (2^4)	(i) 1

11. (a) $\binom{2+2-1}{2}\binom{2+2-1}{2} = \binom{3}{2}\binom{3}{2} = 9$

(b) $\binom{3+2-1}{2}\binom{2+2-1}{2} = \binom{4}{2}\binom{3}{2} = 18$

(c) $\binom{4+2-1}{2}\binom{2+2-1}{2} = \binom{5}{2}\binom{3}{2} = 30$

(d) $\binom{4+2-1}{2}\binom{3+2-1}{2} = \binom{5}{2}\binom{4}{2} = 60$

(e) $\binom{2+2-1}{2}^4 = \binom{3}{2}^4 = 3^4 = 81$

(f) Since $13,860 = 2^2 \cdot 3^2 \cdot 5 \cdot 7 \cdot 11$, it follows that $\mathcal{R}$ contains $\binom{3+2-1}{2}^2\binom{2+2-1}{2}^3 = \binom{4}{2}^2\binom{3}{2}^3 = (36)(27) = 972$ ordered pairs.

12.

$$\begin{aligned}\text{Since } 5880 &= \binom{6+2-1}{2}\binom{4+2-1}{2}\binom{(k+1)+2-1}{2}\\ &= \binom{7}{2}\binom{5}{2}\binom{k+2}{2} = (21)(10)(\tfrac{1}{2})(k+2)(k+1),\\ \text{we find that } 56 &= (k+2)(k+1) \text{ and } k = 6.\end{aligned}$$

For $n = p_1^5 p_2^3 p_3^6$ there are $(5+1)(3+1)(6+1) = (6)(4)(7) = 168$ positive integer divisors, so $|A| = 168$.

13. There may exist an element $a \in A$ such that for all $b \in B$ neither (a, b) nor $(b, a) \in \mathcal{R}$.

14. There are n ordered pairs of the form $(x, x), x \in A$. For each of the $(n^2 - n)/2$ sets $\{(x, y), (y, x)\}$ of ordered pairs where $x, y \in A, x \neq y$, one element is chosen. This results in a maximum value of $n + (n^2 - n)/2 = (n^2 + n)/2$.

The number of antisymmetric relations that can have this size is $2^{(n^2-n)/2}$.

15. $r - n$ counts the elements in $\mathcal{R}$ of the form $(a, b), a \neq b$. Since $\mathcal{R}$ is symmetric, $r - n$ is even.

16. (a) $x\mathcal{R}y$ if $x < y$.

(b) For example, suppose that $\mathcal{R}$ satisfies conditions (ii) and (iii). Since $\mathcal{R} \neq \emptyset$, let $(x, y) \in \mathcal{R}$, for $x, y \in A$. Since $\mathcal{R}$ is symmetric, it follows that $(y, x) \in \mathcal{R}$. Then by the transitive property we have $(x, x) \in \mathcal{R}$ (and $(y, y) \in \mathcal{R}$). But if $(x, x) \in \mathcal{R}$ the relation $\mathcal{R}$ is *not* irreflexive.

(c) $2^{(n^2-n)}$; $2^{n^2} - 2(2^{(n^2-n)})$

17. (a) $\binom{7}{5}\binom{21}{0} + \binom{7}{3}\binom{21}{1} + \binom{7}{0}\binom{21}{2}$

(b) $\binom{7}{5}\binom{21}{0} + \binom{7}{3}\binom{21}{1} + \binom{7}{1}\binom{21}{2}$

18. (a) Let $A_1 = f^{-1}(x)$, $A_2 = f^{-1}(y)$, and $A_3 = f^{-1}(z)$. Then $\mathcal{R} = (A_1 \times A_1) \cup (A_2 \times A_2) \cup (A_3 \times A_3)$, so $|\mathcal{R}| = 10^2 + 10^2 + 5^2 = 225$.

(b) $n_1^2 + n_2^2 + n_3^2 + n_4^2$

Section 7.2

1. $\mathcal{R} \circ \mathcal{S} = \{(1,3), (1,4)\}$; $\mathcal{S} \circ \mathcal{R} = \{(1,2), (1,3), (1,4), (2,4)\}$;
$\mathcal{R}^2 = \mathcal{R}^3 = \{(1,4), (2,4), (4,4)\}$;
$\mathcal{S}^2 = \mathcal{S}^3 = \{(1,1), (1,2), (1,3), (1,4)\}$.

2. Let $x \in A$. $\mathcal{R}$ reflexive $\Longrightarrow (x, x) \in \mathcal{R}$. $(x, x) \in \mathcal{R}, (x, x) \in \mathcal{R} \Longrightarrow (x, x) \in \mathcal{R} \circ \mathcal{R} = \mathcal{R}^2$.

3. $(a, d) \in (\mathcal{R}_1 \circ \mathcal{R}_2) \circ \mathcal{R}_3 \Longrightarrow (a, c) \in \mathcal{R}_1 \circ \mathcal{R}_2, (c, d) \in \mathcal{R}_3$ for some $c \in C \Longrightarrow (a, b) \in \mathcal{R}_1, (b, c) \in \mathcal{R}_2, (c, d) \in \mathcal{R}_3$ for some $b \in B, c \in C \Longrightarrow (a, b) \in \mathcal{R}_1, (b, d) \in \mathcal{R}_2 \circ \mathcal{R}_3 \Longrightarrow (a, d) \in \mathcal{R}_1 \circ (\mathcal{R}_2 \circ \mathcal{R}_3)$, and $(\mathcal{R}_1 \circ \mathcal{R}_2) \circ \mathcal{R}_3 \subseteq \mathcal{R}_1 \circ (\mathcal{R}_2 \circ \mathcal{R}_3)$.

4. (a) $\mathcal{R}_1 \circ (\mathcal{R}_2 \cup \mathcal{R}_3) = \mathcal{R}_1 \circ \{(w,4),(w,5),(x,6),(y,4),(y,5),(y,6)\}$
$= \{(1,4),(1,5),(3,4),(3,5),(2,6),(1,6)\}$
$(\mathcal{R}_1 \circ \mathcal{R}_2) \cup (\mathcal{R}_1 \circ \mathcal{R}_3)$
$= \{(1,5),(3,5),(2,6),(1,4),(1,6)\} \cup \{(1,4),(1,5),(3,4),(3,5)\}$
$= \{(1,4),(1,5),(1,6),(2,6),(3,4),(3,5)\}$

(b) $\mathcal{R}_1 \circ (\mathcal{R}_2 \cap \mathcal{R}_3) = \mathcal{R}_1 \circ \{(w,5)\} = \{(1,5),(3,5)\}$
$(\mathcal{R}_1 \circ \mathcal{R}_2) \cap (\mathcal{R}_1 \circ \mathcal{R}_3) = \{(1,5),(3,5),(2,6),(1,4),(1,6)\} \cap \{(1,4),(1,5),(3,4),(3,5)\} = \{(1,4),(1,5),(3,5)\}$.

5. $\mathcal{R}_1 \circ (\mathcal{R}_2 \cap \mathcal{R}_3) = \mathcal{R}_2 \circ \{(m,3),(m,4)\} = \{(1,3),(1,4)\}$
$(\mathcal{R}_1 \circ \mathcal{R}_2) \cap (\mathcal{R}_1 \circ \mathcal{R}_3) = \{(1,3),(1,4)\} \cap \{(1,3),(1,4)\} = \{(1,3),(1,4)\}$.

6. (a) $(x,z) \in \mathcal{R}_1 \circ (\mathcal{R}_2 \cup \mathcal{R}_3) \iff$ for some $y \in B, (x,y) \in \mathcal{R}_1, (y,z) \in \mathcal{R}_2 \cup \mathcal{R}_3 \iff$ for some $y \in B, ((x,y) \in \mathcal{R}_1, (y,z) \in \mathcal{R}_2)$ or $((x,y) \in \mathcal{R}_1, (y,z) \in \mathcal{R}_3) \Longrightarrow (x,z) \in \mathcal{R}_1 \circ \mathcal{R}_2$ or $(x,z) \in \mathcal{R}_1 \circ \mathcal{R}_3 \iff (x,z) \in (\mathcal{R}_1 \circ \mathcal{R}_2) \cup (\mathcal{R}_1 \circ \mathcal{R}_3)$, so $\mathcal{R}_1 \circ (\mathcal{R}_2 \circ \mathcal{R}_3) \subseteq (\mathcal{R}_1 \circ \mathcal{R}_2) \cup (\mathcal{R}_1 \circ \mathcal{R}_3)$. For the opposite inclusion, $(x,z) \in (\mathcal{R}_1 \circ \mathcal{R}_2) \cup (\mathcal{R}_1 \circ \mathcal{R}_3) \Longrightarrow (x,z) \in \mathcal{R}_1 \circ \mathcal{R}_2$ or $(x,z) \in \mathcal{R}_1 \circ \mathcal{R}_3$. Assume without loss of generality that $(x,z) \in \mathcal{R}_1 \circ \mathcal{R}_2$. Then there exists an element $y \in B$ so that $(x,y) \in \mathcal{R}_1$ and $(y,z) \in \mathcal{R}_2$. But $(y,z) \in \mathcal{R}_2 \Longrightarrow (y,z) \in \mathcal{R}_2 \cup \mathcal{R}_3$, so $(x,z) \in \mathcal{R}_1 \circ (\mathcal{R}_2 \cup \mathcal{R}_3)$, and the result follows.

(b) The proof here is similar to that in part (a). To show that the inclusion can be proper, let $A = B = C = \{1,2,3\}$ with $\mathcal{R}_1 = \{(1,2),(1,1)\}, \mathcal{R}_2 = \{(2,3)\}, \mathcal{R}_3 = \{(1,3)\}$. Then $\mathcal{R}_1 \circ (\mathcal{R}_2 \circ \mathcal{R}_3) = \mathcal{R}_1 \circ \emptyset = \emptyset$, but $(\mathcal{R}_1 \circ \mathcal{R}_2) \circ (\mathcal{R}_1 \circ \mathcal{R}_3) = \{(1,3)\}$.

7. This follows by the Pigeonhole Principle. Here the pigeons are the $2^{n^2}+1$ integers between 0 and 2^{n^2}, inclusive, and the pigeonholes are the 2^{n^2} relations on A.

8. Let $S = \{(1,1),(1,2),(1,4)\}$ and $T = \{(2,1),(2,2),(1,4)\}$.

9. Here there are two choices for each $a_{ii}, 1 \leq i < 6$. For each pair $a_{ij}, a_{ji}, 1 \leq i < j \leq 6$, there are two choices, and there are $(36-6)/2 = 15$ such pairs. Consequently there are $(2^6)(2^{15}) = 2^{21}$ such matrices.

10. For each 0 in E the matrix F can have either 0 or 1 (the other entries in F are 1). Since there are seven 0's in E there are 2^7 possible matrices F. There are 2^5 possible matrices G.

11. Consider the entry in the i-th row and j-th column of $M(\mathcal{R}_1 \circ \mathcal{R}_2)$. If this entry is a 1 then there exists $b_k \in B$ where $1 \leq k \leq n$ and $(a_i, b_k) \in \mathcal{R}_1, (b_k, c_j) \in \mathcal{R}_2$. Consequently, the entry in the i-th row and k-th column of $M(\mathcal{R}_1)$ is 1 and the entry in the k-th row and j-th column of $M(\mathcal{R}_2)$ is 1. This results in a 1 in the i-th row and j-th column in the product $M(\mathcal{R}_1) \cdot M(\mathcal{R}_2)$.

Should the entry in row i and column j of $M(\mathcal{R}_1 \circ \mathcal{R}_2)$ be 0, then for each $b_k, 1 \leq k \leq n$, either $(a_i, b_k) \notin \mathcal{R}_1$ or $(b_k, c_j) \notin \mathcal{R}_2$. This means that in the matrices $M(\mathcal{R}_1), M(\mathcal{R}_2)$, if the entry in the i-th row and k-th column of $M(\mathcal{R}_1)$ is 1 then the entry in the k-th row and j-th column of $M(\mathcal{R}_2)$ is 0. Hence the entry in the i-th row and j-th column of $M(\mathcal{R}_1) \cdot M(\mathcal{R}_2)$ is 0.

12. (a) If $M(\mathcal{R}) = \mathbf{0}$, then $\forall x, y \in A \ \ (x, y) \notin \mathcal{R}$. Hence $\mathcal{R} = \emptyset$. Conversely, if $M(\mathcal{R}) \neq \mathbf{0}$, then $\exists x, y \in A$ where $x\mathcal{R}y$. Hence $(x, y) \in \mathcal{R}$ and $\mathcal{R} \neq \emptyset$.

(c) For $m = 1$, we have $M(\mathcal{R}^1) = M(\mathcal{R}) = [M(\mathcal{R})]^1$, so the result is true in this case. Assuming the truth of the statement for $m = k$ we have $M(\mathcal{R}^k) = [M(\mathcal{R})]^k$. Now consider $m = k + 1$. $M(\mathcal{R}^{k+1}) = M(\mathcal{R} \circ \mathcal{R}^k) = M(\mathcal{R}) \cdot M(\mathcal{R}^k)$ (from Exercise 11) $= M(\mathcal{R}) \cdot [M(\mathcal{R})]^k = [M(\mathcal{R})]^{k+1}$. Consequently this result is true for all $m \geq 1$ by the Principle of Mathematical Induction.

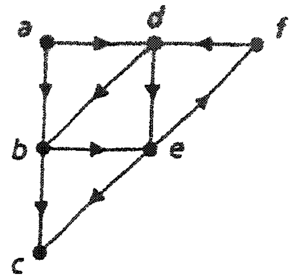
13. (a) $\mathcal{R}$ reflexive $\iff (x, x) \in \mathcal{R}$, for all $x \in A \iff m_{xx} = 1$ in $M = (m_{ij})_{n \times n}$, for all $x \in A \iff I_n \leq M$.

(b) $\mathcal{R}$ symmetric $\iff [\forall x, y \in A \ \ (x, y) \in \mathcal{R} \implies (y, x) \in \mathcal{R}] \iff [\forall x, y \in A \ \ m_{xy} = 1$ in $M \implies m_{yx} = 1$ in $M] \iff M = M^{tr}$.

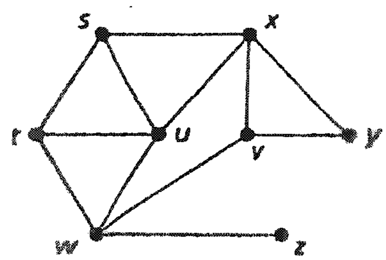
14.

```
10!     THIS PROGRAM MAY BE USED TO DETERMINE IF A RELATION
20!     ON A SET OF SIZE N, WHERE N <= 20, IS AN
30!     EQUIVALENCE RELATION. WE ASSUME WITHOUT LOSS OF
40!     GENERALITY THAT THE ELEMENTS ARE 1,2,3,...,N.
50!
60      INPUT "N ="; N
70      PRINT "   INPUT THE RELATION MATRIX FOR THE RELATION"
80      PRINT "BEING EXAMINED BY TYPING A(I,J) = 1 FOR EACH"
90      PRINT "1 <=  I <= N, 1 <=  J <= N, WHERE (I,J) IS IN"
100     PRINT "THE RELATION. WHEN ALL THE ORDERED PAIRS HAVE"
110     PRINT "BEEN ENTERED TYPE 'CONT' "
120     STOP
130     DIM A(20,20), C(20,20), D(20,20)
140     FOR K = 1 TO N
150         T = T + A(K,K)
160     NEXT K
170     IF T = N THEN &
                PRINT "R IS REFLEXIVE"; X = 1: GO TO 190
180     PRINT "R IS NOT REFLEXIVE"
190     FOR I = 1 TO N
200         FOR J = I + 1 TO N
210             IF A(I,J) <> A(J,I) THEN GO TO 260
220         NEXT J
230     NEXT I
240     PRINT "R IS SYMMETRIC": Y = 1
250     GO TO 270
260     PRINT "R IS NOT SYMMETRIC"
270     MAT C = A
280     MAT D = A*C
290     FOR I = 1 TO N
300         FOR J = 1 TO N
310             IF D(I,J) > 0 AND A(I,J) = 0 THEN GO TO 360
320         NEXT J
330     NEXT I
340     PRINT "R IS TRANSITIVE"; Z = 1
350     GO TO 370
360     PRINT "R IS NOT TRANSITIVE"
370     IF X + Y + Z = 3 THEN &
                    PRINT "R IS AN EQUIVALENCE RELATION" &
        ELSE PRINT "R IS NOT AN EQUIVALENCE RELATION"
380     END
```

15. (a)



(b)



16. (a) True (b) True (c) True (d) False

17. (i) $\mathcal{R} = \{(a,b),(b,a),(a,e),(e,a),(b,c),(c,b),(b,d),(d,b),(b,e),(e,b),(d,e),(e,d),(d,f),(f,d)\}$

$$M(\mathcal{R}) = \begin{array}{c} \\ (a) \\ (b) \\ (c) \\ (d) \\ (e) \\ (f) \end{array} \begin{array}{c} (a)\,(b)\,(c)\,(d)\,(e)\,(f) \\ \begin{bmatrix} 0 & 1 & 0 & 0 & 1 & 0 \\ 1 & 0 & 1 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \end{bmatrix} \end{array}$$

For parts (ii), (iii), and (iv), the rows and columns of the relation matrix are indexed as in part (i).

(ii) $\mathcal{R} = \{(a,b),(b,e),(d,b),(d,c),(e,f)\}$

$$M(\mathcal{R}) = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

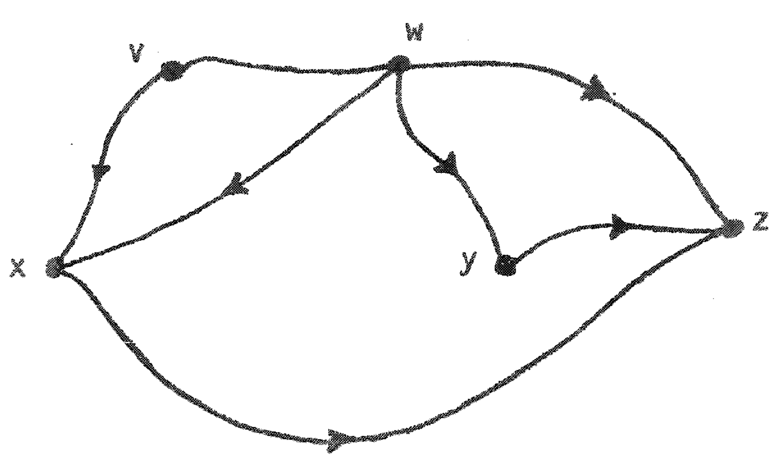
(iii) $\mathcal{R} = \{(a,a),(a,b),(b,a),(c,d),(d,c),(d,e),(e,d),(d,f),(f,d),(e,f),(f,e)\}$

$$M(\mathcal{R}) = \begin{bmatrix} 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 & 0 \end{bmatrix}$$

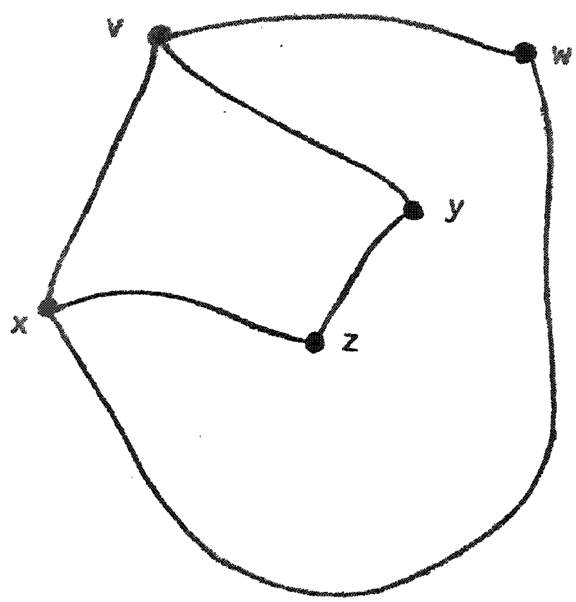
(iv) $\mathcal{R} = \{(b,a),(b,c),(c,b),(b,e),(c,d),(e,d)\}$

$$M(\mathcal{R}) = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

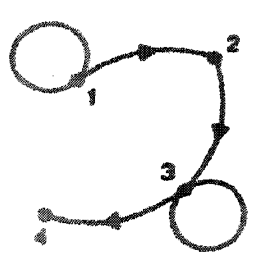
18. (a) $\mathcal{R} = \{(v,w),(v,x),(w,v),(w,x),(w,y),(w,z),(x,z),(y,z)\}$



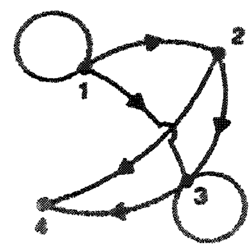
(b) $\mathcal{R} = \{(v,w),(v,x),(v,y),(w,v),(w,x),(x,v),(x,w),(x,z),(y,v),(y,z),(z,x),(z,y)\}$



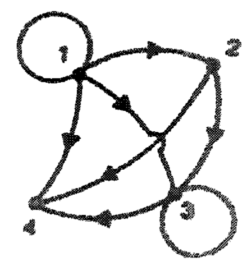
19. $\mathcal{R}$:



$\mathcal{R}^2$:



$\mathcal{R}^3$ and $\mathcal{R}^4$:



20. (a) (i) $\binom{7}{2}$

(ii) Each directed path corresponds to a subset of $\{2,3,4,5,6\}$. There are 2^5 subsets of $\{2,3,4,5,6\}$ and, consequently, 2^5 directed paths in G from 1 to 7.

(b) (i) $\binom{n}{2} = |E|$.

(ii) There are 2^{n-2} directed paths in G from 1 to n.

(iii) There are $2^{[((b-a)+1)-2]} = 2^{b-a-1}$ directed paths in G from a to b.

21. 2^{25}; $(2^5)(2^{10}) = 2^{15}$

22. 2^{25}; $(2^5)(2^{10}) = 2^{15}$

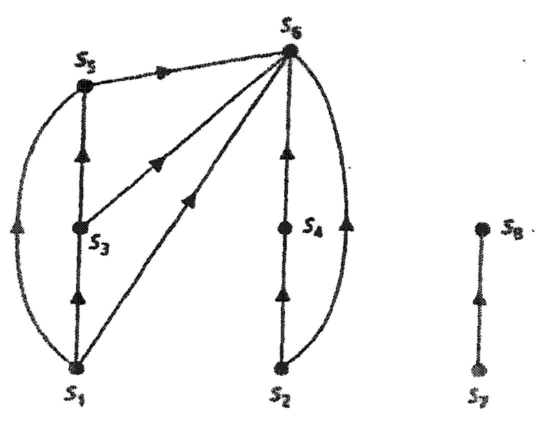
23. (a) $\mathcal{R}_1$: $\qquad\qquad$ $\mathcal{R}_2$:

$$\begin{bmatrix} 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \qquad \begin{bmatrix} 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 \end{bmatrix}$$

(b) Given an equivalence relation $\mathcal{R}$ on a finite set A, list the elements of A so that elements in the same cell of the partition (See Section 7.4.) are adjacent. The resulting relation matrix will then have square blocks of 1's along the diagonal (from upper left to lower right).

24. $\binom{6}{2}$; $\binom{7}{2}$; $\binom{n}{2}$

25.



26. (a) Let $k \in \mathbf{Z}^+$. Then $\mathcal{R}^{12k} = \{(1,1),(2,2),(3,3),(4,4),(5,5),(6,6),(7,7)\}$ and $\mathcal{R}^{12k+1} = \mathcal{R}$. The smallest value of $n > 1$ such that $\mathcal{R}^n = \mathcal{R}$ is $n = 13$. For all multiples of 12 the graph consists of all loops. When $n = 3, (5,5),(6,6),(7,7) \in \mathcal{R}^3$, and this is the smallest power of $\mathcal{R}$ that contains at least one loop.

(b) When $n = 2$, we find $(1,1),(2,2)$ in $\mathcal{R}$. For all $k \in \mathbf{Z}^+, \mathcal{R}^{30k} = \{(x,x) | x \in \mathbf{Z}^+, 1 \leq x \leq 10\}$ and $\mathcal{R}^{30k+1} = \mathcal{R}$. Hence $\mathcal{R}^{31}$ is the smallest power of $\mathcal{R}$ (for $n > 1$) where $\mathcal{R}^n = \mathcal{R}$.

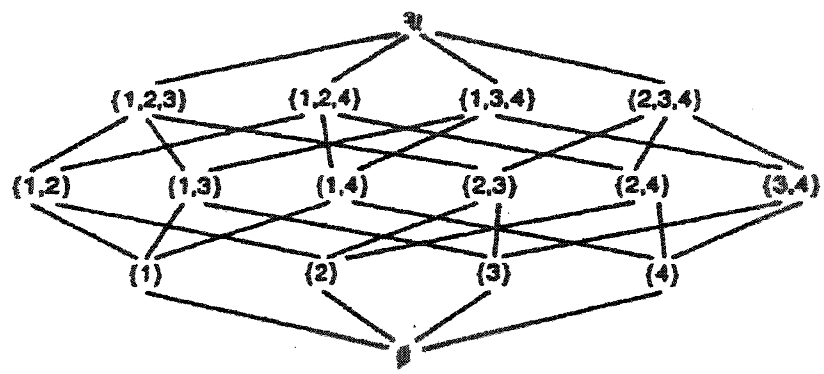
(c) Let $\mathcal{R}$ be a relation on set A where $|A| = m$. Let G be the directed graph associated with $\mathcal{R}$ – each component of G is a directed cycle C_i on m_i vertices, with $1 \leq i \leq k$. (Thus $m_1 + m_2 + \ldots + m_k = m$.) The smallest power of $\mathcal{R}$ where loops appear is $\mathcal{R}^t$, for $t = min\{m_i | 1 \leq i \leq k\}$.

Let $s = \text{lcm}(m_1, m_2, \ldots, m_k)$. Then $\mathcal{R}^{rs}$ = the identity (equality) relation on A and $\mathcal{R}^{rs+1} = \mathcal{R}$, for all $r \in \mathbf{Z}^+$. The smallest power of $\mathcal{R}$ that reproduces $\mathcal{R}$ is $s + 1$.

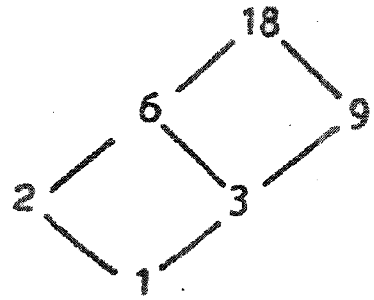
27. $\binom{n}{2} = 703 \Rightarrow n = 38$

Section 7.3

1.



2.



3. For all $a \in A, b \in B, a\mathcal{R}_1 a$ and $b\mathcal{R}_2 b$ so $(a,b)\mathcal{R}(a,b)$, and $\mathcal{R}$ is reflexive. Next $(a,b)\mathcal{R}(c,d), (c,d)\mathcal{R}(a,b) \implies a\mathcal{R}_1 c, c\mathcal{R}_1 a$ and $b\mathcal{R}_2 d, d\mathcal{R}_2 b \implies a = c, b = d \implies (a,b) = (c,d)$, so $\mathcal{R}$ is antisymmetric. Finally, $(a,b)\mathcal{R}(c,d), (c,d)\mathcal{R}(e,f) \implies a\mathcal{R}_1 c, c\mathcal{R}_1 e$ and $b\mathcal{R}_2 d, d\mathcal{R}_2 f \implies a\mathcal{R}_1 e, b\mathcal{R}_2 f \implies (a,b)\mathcal{R}(e,f)$, and $\mathcal{R}$ is transitive. Consequently, $\mathcal{R}$ is a partial order.

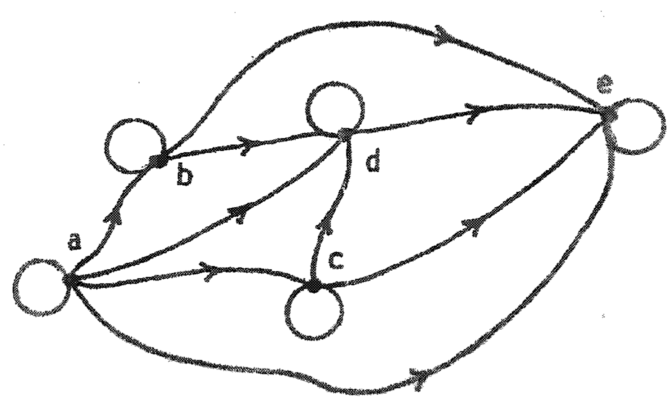
4. No. Let $A = B = \{1,2\}$ with each of $\mathcal{R}_1, \mathcal{R}_2$ the usual "is less than or equal to" relation. Then $\mathcal{R}$ is a partial order but it is not a total order for we cannot compare (1,2) and (2,1).

5. $\emptyset < \{1\} < \{2\} < \{3\} < \{1,2\} < \{1,3\} < \{2,3\} < \{1,2,3\}$. (There are other possibilities.)

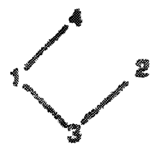
6. (a)

$$M(\mathcal{R}) = \begin{array}{c} \\ (a) \\ (b) \\ (c) \\ (d) \\ (e) \end{array} \begin{array}{c} \begin{array}{ccccc} (a) & (b) & (c) & (d) & (e) \end{array} \\ \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \end{array}$$

(b)



(c) $a < b < c < d < e$ or $a < c < b < d < e$

7. (a) 

(b) $3 < 2 < 1 < 4$ or $3 < 1 < 2 < 4$.

(c) 2

8. Suppose that $x, y \in A$ and that both are least elements. Then $x\mathcal{R}y$ since x is a least element, and $y\mathcal{R}x$ since y is a least element. With $\mathcal{R}$ antisymmetric we have $x = y$.

9. Let x, y both be greatest lower bounds. Then $x\mathcal{R}y$ since x is a lower bound and y is a greatest lower bound. By similar reasoning $y\mathcal{R}x$. Since $\mathcal{R}$ is antisymmetric, $x = y$. [The proof for the *lub* is similar.]

10. Let $\mathcal{U} = \{1, 2, 3, 4\}$. Let A be the collection of all proper subsets of $\mathcal{U}$, partially ordered under set inclusion. Then {1,2,3}, {1,2,4}, {1,3,4}, and {2,3,4} are all maximal elements.

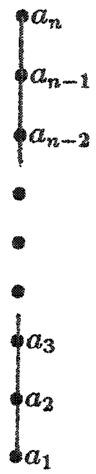
11. Let $\mathcal{U} = \{1, 2\}, A = \mathcal{P}(\mathcal{U})$, and $\mathcal{R}$ the inclusion relation. Then $(A, \mathcal{R})$ is a poset but not a total order. Let $B = \{\emptyset, \{1\}\}$. Then $(B \times B) \cap \mathcal{R}$ is a total order.

12. For all vertices $x, y \in A, x \neq y$, there is either an edge (x, y) or an edge (y, x), but not both. In addition, if $(x, y), (y, z)$ are edges in G then (x, z) is an edge in G. Finally, at every vertex of the graph there is a loop.

13. $n + \binom{n}{2}$

14. $n + \binom{n}{2}$

15. (a) The n elements of A are arranged along a vertical line. For if $A = \{a_1, a_2, \ldots a_n\}$, where $a_1\mathcal{R}a_2\mathcal{R}a_3\mathcal{R}\ldots\mathcal{R}a_n$, then the diagram can be drawn as



(b) $n!$

16. (a) Let $a \in A$ with a minimal. Then for $x \in A$, $x\mathcal{R}a \Longrightarrow x = a$. So if $M(\mathcal{R})$ is the relation matrix for $\mathcal{R}$, the column under 'a' has all 0's except for the one 1 for the ordered pair (a, a).

(b) Let $b \in A$, with b a greatest element. Then the column under 'b' in $M(\mathcal{R})$ has all 1's. If $c \in A$ and c is a least element, then the row of $M(\mathcal{R})$ determined by 'c' has all 1's.

17.

	lub	*glb*		*lub*	*glb*		*lub*	*glb*
(a)	{1,2}	$\emptyset$	(c)	{1,2}	$\emptyset$	(e)	{1,2,3}	$\emptyset$
(b)	{1,2,3}	$\emptyset$	(d)	{1,2,3}	{1}			

18. (a) (i) Only one such upper bound – {1,2,3}. (ii) Here the upper bound has the form $\{1, 2, 3, x\}$ where $x \in \mathcal{U}$ and $4 \leq x \leq 7$. Hence there are four such upper bounds. (iii) There are $\binom{4}{2}$ upper bounds of B that contain five elements from $\mathcal{U}$.

(b) $\binom{4}{0} + \binom{4}{1} + \binom{4}{2} + \binom{4}{3} + \binom{4}{4} = 2^4 = 16$

(c) $lub\, B = \{1, 2, 3\}$

(d) One – namely $\emptyset$

(e) $glb\, B = \emptyset$

19. For each $a \in \mathbf{Z}$ it follows that $a\mathcal{R}a$ because $a - a = 0$, an even nonnegative integer. Hence $\mathcal{R}$ is *reflexive.* If $a, b, c \in \mathbf{Z}$ with $a\mathcal{R}b$ and $b\mathcal{R}c$ then

$$a - b = 2m, \text{ for some } m \in \mathrm{N}$$
$$b - c = 2n, \text{ for some } n \in \mathrm{N},$$

and $a - c = (a - b) + (b - c) = 2(m + n)$, where $m + n \in \mathrm{N}$. Therefore, $a\mathcal{R}c$ and $\mathcal{R}$ is *transitive.* Finally, suppose that $a\mathcal{R}b$ and $b\mathcal{R}a$ for some $a, b \in \mathbf{Z}$. Then $a - b$ and $b - a$ are both nonnegative integers. Since this can only occur for $a - b = b - a$, we find that $[a\mathcal{R}b \wedge b\mathcal{R}a] \Rightarrow a = b$, so $\mathcal{R}$ is *antisymmetric.*

Consequently, the relation $\mathcal{R}$ is a partial order for $\mathbf{Z}$. But it is *not* a total order. For example, $2, 3 \in \mathbf{Z}$ and we have neither $2\mathcal{R}3$ nor $3\mathcal{R}2$, because neither -1 nor 1, respectively, is a nonnegative even integer.

20. (a) For all $(a,b) \in A$, $a = a$ and $b \leq b$, so $(a,b)\mathcal{R}(a,b)$ and the relation is reflexive. If $(a,b),(c,d) \in A$ with $(a,b)\mathcal{R}(c,d)$ and $(c,d)\mathcal{R}(a,b)$, then if $a \neq c$ we find that

$(a,b)\mathcal{R}(c,d) \Rightarrow a < c$, and

$(c,d)\mathcal{R}(a,b) \Rightarrow c < a$,

and we obtain $a < a$. Hence we have $a = c$.

And now we find that

$(a,b)\mathcal{R}(c,d) \Rightarrow b \leq d$, and

$(c,d)\mathcal{R}(a,b) \Rightarrow d \leq b$,

so $b = d$. Therefore, $(a,b)\mathcal{R}(c,d)$ and $(c,d)\mathcal{R}(a,b) \Rightarrow (a,b) = (c,d)$, so the relation is antisymmetric. Finally, consider $(a,b),(c,d),(e,f) \in A$ with $(a,b)\mathcal{R}(c,d)$ and $(c,d)\mathcal{R}(e,f)$. Then

(i) $a < c$, or (ii) $a = c$ and $b \leq d$; and

(i)′ $c < e$, or (ii)′ $c = e$ and $d \leq f$.

Consequently,

(i)″ $a < e$ or (ii)″ $a = e$ and $b \leq f$ — so, $(a,b)\mathcal{R}(e,f)$ and the relation is transitive.

The preceding shows that $\mathcal{R}$ is a partial order on A.

b) & c) There is only one minimal element — namely, (0,0). This is also the least element for this partial order.

The element (1,1) is the only maximal element for the partial order. It is also the greatest element.

d) This partial order is a total order. We find here that

$$(0,0)\mathcal{R}(0,1)\mathcal{R}(1,0)\mathcal{R}(1,1).$$

21. (a) The reflexive, antisymmetric, and transitive properties are established as in the previous exercise.

(b) & (c) Here the least element (and only minimal element) is (0,0). The element (2,2) is the greatest element (and the only maximal element).

(d) Once again we obtain a total order, for

$$(0,0)\mathcal{R}(0,1)\mathcal{R}(0,2)\mathcal{R}(1,0)\mathcal{R}(1,1)\mathcal{R}(1,2)\mathcal{R}(2,0)\mathcal{R}(2,1)\mathcal{R}(2,2).$$

22. Here $|X| = n+1$, $|A| = (n+1)^2$ and $|\mathcal{R}| = (n+1)^2 + \binom{(n+1)^2}{2}$.

23. (a) False. Let $\mathcal{U} = \{1,2\}, A = \mathcal{P}(\mathcal{U})$, and $\mathcal{R}$ be the inclusion relation. Then $(A,\mathcal{R})$ is a lattice where for all $S,T \in A$, $lub\{S,T\} = S \cup T$ and $glb\{S,T\} = S \cap T$. However, $\{1\}$ and $\{2\}$ are not related, so $(A,\mathcal{R})$ is not a total order.

(b) If $(A, \mathcal{R})$ is a total order, then for all $x, y \in A, x\mathcal{R}y$ or $y\mathcal{R}x$. For $x\mathcal{R}y, lub\{x, y\} = y$ and $glb\{x, y\} = x$. Consequently, $(A, \mathcal{R})$ is a lattice.

24. Since A is finite, A has a maximal element, by Theorem 7.3. If x, y $(x \neq y)$ are both maximal elements, since $x, y\mathcal{R}lub\{x, y\}$, then $lub\{x, y\}$ must equal either x or y. Assume $lub\{x, y\} = x$. Then $y\mathcal{R}x$, so y cannot be a maximal element. Hence A has a unique maximal element x. Now for each $a \in A, a \neq x$, if $lub\{a, x\} \neq x$, then we contradict x being a maximal element. Hence $a\mathcal{R}x$ for all $a \in A$, so x is the greatest element in A. [The proof for the least element is similar.]

25. (a) a (b) a (c) c (d) e (e) z (f) e (g) v
$(A, \mathcal{R})$ is a lattice with z the greatest (and only maximal) element and a the least (and only minimal) element.

26.

a)	5	b) and c)	$n + 1$
d)	10	e) and f)	$n + (n-1) + \cdots + 2 + 1 = n(n+1)/2.$

27. Consider the vertex $p^a q^b r^c$, $0 \leq a < m$, $0 \leq b < n$, $0 \leq c < k$. There are mnk such vertices; each determines three edges — going to the vertices $p^{a+1}q^b r^c, p^a q^{b+1} r^c, p^a q^b r^{c+1}$. This accounts for $3mnk$ edges.

Now consider the vertex $p^m q^b r^c$, $0 \leq b < n$, $0 \leq c < k$. There are nk of these vertices; each detemines two edges — going to the vertices $p^m q^{b+1} r^c$, $p^m q^b r^{c+1}$. This accounts for $2nk$ edges. And similar arguments for the vertices $p^a q^n r^c (0 \leq a < m, 0 \leq c < k)$ and $p^a q^b r^k (0 \leq a < m, 0 \leq b < n)$ account for $2mk$ and $2mn$ edges, respectively.

Finally, each of the k vertices $p^m q^n r^c$, $0 \leq c < k$, determines one edge (going to $p^m q^n r^{c+1}$) and so these vertices account for k new edges. Likewise, each of the n vertices $p^m q^b r^k$, $0 \leq b < n$, determines one edge (going to $p^m q^{b+1} r^k$), and so these vertices account for n new edges. Lastly, each of the m vertices $p^a q^n r^k$, $0 \leq a < m$, determines one edge (going to $p^{a+1} q^n r^k$) and these vertices account for m new edges.

The preceding results give the total number of edges as $(m+n+k)+2(mn+mk+nk)+3mnk$.

28. a) $24 = 2^3 \cdot 3$. There are $4 \cdot 2 = 8$ divisors for this partial order and they can be totally ordered in $\frac{1}{5}\binom{8}{4} = 14$ ways.
b) $75 = 3 \cdot 5^2$. There are $2 \cdot 3 = 6$ divisors for this partial order and they can be totally ordered in $\frac{1}{4}\binom{6}{3} = 5$ ways.
c) $1701 = 3^5 \cdot 7$. Here the 12 divisors can be totally ordered in $\frac{1}{7}\binom{12}{6} = 132$ ways.

29. $429 = (\frac{1}{8})\binom{14}{7}$ so $k = 6$, and there are $2 \cdot 7 = 14$ positive integer divisors of $p^6 q$.

30. For the (0,1)-matrix $E = (e_{ij})_{m \times n}$ we have $e_{ij} = e_{ij}$, so $e_{ij} \leq e_{ij}$, for all $1 \leq i \leq m$, $1 \leq j \leq n$. Consequently, $E \leq E$ and the "precedes" relation is reflexive.

Now let $E = (e_{ij})_{m\times n}$, $F = (f_{ij})_{m\times n}$ be $(0,1)$-matrices, with $E \le F$ and $F \le E$. Then, for all $1 \le i \le m$, $1 \le j \le n$, $e_{ij} \le f_{ij}$ and $f_{ij} \le e_{ij} \Rightarrow e_{ij} = f_{ij}$, so $E = F$ – and the "precedes" relation is antisymmetric.

Finally, suppose that $E = (e_{ij})_{m\times n}$, $F = (f_{ij})_{m\times n}$, and $G = (g_{ij})_{m\times n}$ are $(0,1)$-matrices, with $E \le F$ and $F \le G$. Then, for all $1 \le i \le m$, $1 \le j \le n$, $e_{ij} \le f_{ij}$ and $f_{ij} \le g_{ij} \Rightarrow e_{ij} \le g_{ij}$, so $E \le G$ – and the "precedes" relation is transitive.

In so much as the "precedes" relation is reflexive, antisymmetric, and transitive, it follows that this relation is a partial order – making A into a poset.

Section 7.4

1. (a) Here the collection A_1, A_2, A_3 provides a partition of A.
(b) Although $A = A_1 \cup A_2 \cup A_3 \cup A_4$, we have $A_1 \cap A_2 \neq \emptyset$, so the collection A_1, A_2, A_3, A_4 does *not* provide a partition for A.

2. (a) There are three choices for placing 8 — in either A_1, A_2, or A_3. Hence there are three partitions of A for the conditions given.
(b) There are two possibilities with $7 \in A_1$, and two others with $8 \in A_1$. Hence there are four partitions of A under these conditions.
(c) If we place 7,8 in the same cell for a partition we obtain three of the possibilities. If not, there are three choices of cells for 7 and two choices of cells for 8 — and six more partitions that satisfy the stated restrictions. In total — by the rules of sum and product — there are $3 + (3)(2) = 3 + 6 = 9$ such partitions.

3. $\mathcal{R} = \{(1,1),(1,2),(2,1),(2,2),(3,3),(3,4),(4,3),(4,4),(5,5)\}$.

4. (a) $[1] = \{1,2\} = [2]; [3] = \{3\}$
(b) $A = \{1,2\} \cup \{3\} \cup \{4,5\} \cup \{6\}$.

5. $\mathcal{R}$ is not transitive since $1\mathcal{R}2$, $2\mathcal{R}3$ but $1 \not\mathcal{R} 3$.

6. (a) For all $(x,y) \in A$, since $x = x$, it follows that $(x,y)\mathcal{R}(x,y)$, so $\mathcal{R}$ is reflexive. If $(x_1,y_1),(x_2,y_2) \in A$ and $(x_1,y_1)\mathcal{R}(x_2,y_2)$, then $x_1 = x_2$, so $x_2 = x_1$ and $(x_2,y_2)\mathcal{R}(x_1,y_1)$. Hence $\mathcal{R}$ is symmetric. Finally, let $(x_1,y_1),(x_2,y_2),(x_3,y_3) \in A$ with $(x_1,y_1)\mathcal{R}(x_2,y_2)$ and $(x_2,y_2)\mathcal{R}(x_3,y_3)$. $(x_1,y_1)\mathcal{R}(x_2,y_2) \Longrightarrow x_1 = x_2; (x_2,y_2)\mathcal{R}(x_3,y_3) \Longrightarrow x_2 = x_3$. With $x_1 = x_2, x_2 = x_3$, it follows that $x_1 = x_3$, so $(x_1,y_1)\mathcal{R}(x_3,y_3)$ and $\mathcal{R}$ is transitive.

(b) Each equivalence class consists of the points on a vertical line. The collection of these vertical lines then provides a partition of the real plane.

7. (a) For all $(x,y) \in A, x + y = x + y \Longrightarrow (x,y)\mathcal{R}(x,y)$.
$(x_1,y_1)\mathcal{R}(x_2,y_2) \Longrightarrow x_1 + y_1 = x_2 + y_2 \Longrightarrow x_2 + y_2 = x_1 + y_1 \Longrightarrow$
$(x_2,y_2)\mathcal{R}(x_1,y_1)$. $(x_1,y_1)\mathcal{R}(x_2,y_2),(x_2,y_2)\mathcal{R}(x_3,y_3) \Longrightarrow$

$x_1 + y_1 = x_2 + y_2, x_2 + y_2 = x_3 + y_3$, so $x_1 + y_1 = x_3 + y_3$ and $(x_1, y_1)\mathcal{R}(x_3, y_3)$. Since $\mathcal{R}$ is reflexive, symmetric and transitive, it is an equivalence relation.

(b) $[(1,3)] = \{(1,3),(2,2),(3,1)\}$;
$[(2,4)] = \{(1,5),(2,4),(3,3),(4,2),(5,1)\}$; $[(1,1)] = \{(1,1)\}$.

(c) $A = \{(1,1)\} \cup \{(1,2),(2,1)\} \cup \{(1,3),(2,2),(3,1)\} \cup$
$\{(1,4),(2,3),(3,2),(4,1)\} \cup \{(1,5),(2,4),(3,3),(4,2),(5,1)\} \cup$
$\{(2,5),(3,4),(4,3),(5,2)\} \cup \{(3,5),(4,4),(5,3)\} \cup \{(4,5),(5,4)\} \cup \{(5,5)\}$.

8. (a) For all $a \in A, a - a = 3 \cdot 0$, so $\mathcal{R}$ is reflexive. For $a, b \in A, a - b = 3c$, for some $c \in \mathbf{Z} \Longrightarrow b - a = 3(-c)$, for $-c \in \mathbf{Z}$, so $a\mathcal{R}b \Longrightarrow b\mathcal{R}a$ and $\mathcal{R}$ is symmetric. If $a, b, c \in A$ and $a\mathcal{R}b, b\mathcal{R}c$, then $a - b = 3m, b - c = 3n$, for some $m, n \in \mathbf{Z} \Longrightarrow (a-b)+(b-c) = 3m + 3n \Longrightarrow a - c = 3(m+n)$, so $a\mathcal{R}c$. Consequently, $\mathcal{R}$ is transitive.

(b) $[1] = [4] = [7] = \{1,4,7\}; [2] = [5] = \{2,5\}; [3] = [6] = \{3,6\}$.
$A = \{1,4,7\} \cup \{2,5\} \cup \{3,6\}$.

9. (a) For all $(a,b) \in A$ we have $ab = ab$, so $(a,b)\mathcal{R}(a,b)$ and $\mathcal{R}$ is reflexive. To see that $\mathcal{R}$ is symmetric, suppose that $(a,b),(c,d) \in A$ and that $(a,b)\mathcal{R}(c,d)$. Then $(a,b)\mathcal{R}(c,d) \Rightarrow ad = bc \Rightarrow cb = da \Rightarrow (c,d)\mathcal{R}(a,b)$, so $\mathcal{R}$ is symmetric. Finally, let $(a,b),(c,d),(e,f) \in A$ with $(a,b)\mathcal{R}(c,d)$ and $(c,d)\mathcal{R}(e,f)$. Then $(a,b)\mathcal{R}(c,d) \Rightarrow ad = bc$ and $(c,d)\mathcal{R}(e,f) \Rightarrow cf = de$, so $adf = bcf = bde$ and since $d \neq 0$, we have $af = be$. But $af = be \Rightarrow (a,b)\mathcal{R}(e,f)$, and consequently $\mathcal{R}$ is transitive.
It follows from the above that $\mathcal{R}$ is an equivalence relation on A.

(b) $[(2,14)] = \{(2,14)\}$
$[(-3,-9)] = \{(-3,-9),(-1,-3),(4,12)\}$
$[(4,8)] = \{(-2,-4),(1,2),(3,6),(4,8)\}$

(c) There are five cells in the partition — in fact,

$$A = [(-4,-20)] \cup [(-3,-9)] \cup [(-2,-4)] \cup [(-1,-11)] \cup [(2,14)].$$

10. (a) For all $X \subseteq A, B \cap X = B \cap X$, so $X\mathcal{R}X$ and $\mathcal{R}$ is reflexive. If $X, Y \subseteq A$, then $X\mathcal{R}Y \Longrightarrow X \cap B = Y \cap B \Longrightarrow Y \cap B = X \cap B \Longrightarrow Y\mathcal{R}X$, so $\mathcal{R}$ is symmetric. And finally, if $W, X, Y \subseteq A$ with $W\mathcal{R}X$ and $X\mathcal{R}Y$, then $W \cap B = X \cap B$ and $X \cap B = Y \cap B$. Hence $W \cap B = Y \cap B$, so $W\mathcal{R}Y$ and $\mathcal{R}$ is transitive. Consequently $\mathcal{R}$ is an equivalence relation on $\mathcal{P}(A)$.

(b) $\{\emptyset, \{3\}\} \cup \{\{1\},\{1,3\}\} \cup \{\{2\},\{2,3\}\} \cup \{\{1,2\},\{1,2,3\}\}$

(c) $[X] = \{\{1,3\},\{1,3,4\},\{1,3,5\},\{1,3,4,5\}\}$

(d) 8 – one for each subset of B.

11. (a) $(\frac{1}{2})\binom{6}{3}$ – The factor $(\frac{1}{2})$ is needed because each selection of size 3 should account for only one such equivalence relation, not two. For example, if $\{a,b,c\}$ is selected we get

the partition $\{a,b,c\} \cup \{d,e,f\}$ that corresponds with an equivalence relation. But the selection $\{d,e,f\}$ gives us the same partition and corresponding equivalence relation.
(b) $\binom{6}{3}[1+3] = 4\binom{6}{3}$ – After selecting 3 of the elements we can partition the remaining 3 in
(i) 1 way into three equivalence classes of size 1; or
(ii) 3 ways into one equivalence class of size 1 and one of size 2.
(c) $\binom{6}{4}[1+1] = 2\binom{6}{4}$
(d) $(\frac{1}{2})\binom{6}{3} + 4\binom{6}{3} + 2\binom{6}{4} + \binom{6}{5} + \binom{6}{6}$

12.
(a) $2^{10} = 1024$
(b) $\sum_{i=1}^{5} S(5,i) = 1+15+25+10+1 = 52$
(c) $1024 - 52 = 972$
(d) $S(5,2) = 15$
(e) $\sum_{i=1}^{4} S(4,i) = 1+7+6+1 = 15$
(f) $\sum_{i=1}^{3} S(3,i) = 1+3+1 = 5$
(g) $\sum_{i=1}^{3} S(3,i) = 1+3+1 = 5$
(h) $(\sum_{i=1}^{3} S(3,i)) - (\sum_{i=1}^{2} S(2,i)) = 3$

13. 300

14. (a) Not possible. With $\mathcal{R}$ reflexive, $|\mathcal{R}| \geq 7$.
(b) $\mathcal{R} = \{(x,x)|x \in \mathbf{Z}, 1 \leq x \leq 7\}$.
(c) Not possible. With $\mathcal{R}$ symmetric, $|\mathcal{R}| - 7$ must be even.
(d) $\mathcal{R} = \{(x,x)|x \in \mathbf{Z}, 1 \leq x \leq 7\} \cup \{(1,2),(2,1)\}$.
(e) $\mathcal{R} = \{(x,x)|x \in \mathbf{Z}, 1 \leq x \leq 7\} \cup \{(1,2),(2,1)\} \cup \{(3,4),(4,3)\}$.
(f) and (h) Not possible with $r - 7$ odd.
(g) and (i) Not possible. See the remark at the end of Section 7.4.

15. Let $\{A_i\}_{i \in I}$ be a partition of a set A. Define $\mathcal{R}$ on A by $x\mathcal{R}y$ if for some $i \in I, x, y \in A_i$. For each $x \in A, x, x \in A_i$ for some $i \in I$, so $x\mathcal{R}x$ and $\mathcal{R}$ is reflexive. $x\mathcal{R}y \Longrightarrow x, y \in A_i$, for some $i \in I \Longrightarrow y, x \in A_i$, for some $i \in I \Longrightarrow y\mathcal{R}x$, so $\mathcal{R}$ is symmetric. If $x\mathcal{R}y$ and $y\mathcal{R}z$, then $x, y \in A_i$ and $y, z \in A_j$ for some $i, j \in I$. Since $A_i \cap A_j$ contains y and $\{A_i\}_{i \in I}$ is a partition, from $A_i \cap A_j = \emptyset$ it follows that $A_i = A_j$, so $i = j$. Hence $x, z \in A_i$, so $x\mathcal{R}z$ and $\mathcal{R}$ is transitive.

16. Let $P = \cup_{i \in I} A_i$ be a partition of A. Then $E = \cup_{i \in I}(A_i \times A_i)$ is an equivalence relation and $f(E) = P$, so f is *onto*.

Now let E_1, E_2 be two equivalence relations on A. If $E_1 \neq E_2$, then there exists $x, y \in A$ where $(x,y) \in E_1$ and $(x,y) \notin E_2$. Hence if $f(E_1) = P_1 = \cup_{i \in I} A_i$ and $f(E_2) = P_2 = \cup_{j \in J} A_j$, then $(x,y) \in E_1 \Longrightarrow x, y \in A_i$, $\exists i \in I$, while $(x,y) \notin E_2 \Longrightarrow$ $\forall j \in J\ (x \notin A_j \vee y \notin A_j)$. Consequently, $P_1 \neq P_2$ and f is *one-to-one*.

17. Proof: Since $\{B_1, B_2, B_3, \ldots, B_n\}$ is a partition of B, we have $B = B_1 \cup B_2 \cup B_3 \cup \ldots \cup B_n$. Therefore $A = f^{-1}(B) = f^{-1}(B_1 \cup \ldots \cup B_n) = f^{-1}(B_1) \cup \ldots \cup f^{-1}(B_n)$ [by generalizing part (b) of Theorem 5.10]. For $1 \leq i < j \leq n$, $f^{-1}(B_i) \cap f^{-1}(B_j) = f^{-1}(B_i \cap B_j) = f^{-1}(\emptyset) = \emptyset$. Consequently, $\{f^{-1}(B_i)|1 \leq i \leq n, f^{-1}(B_i) \neq \emptyset\}$ is a partition of A.

Note: Part (b) of Example 7.55 is a special case of this result.

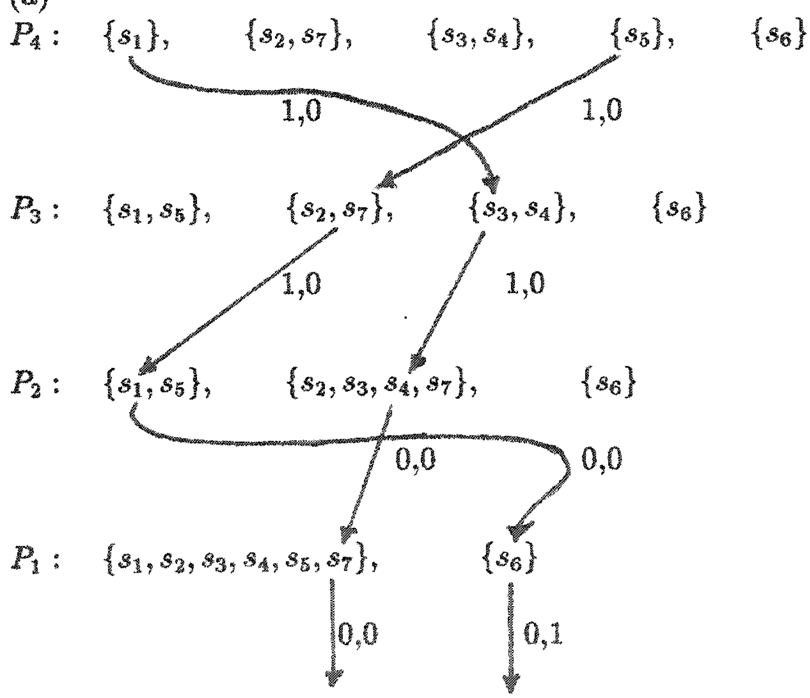
Section 7.5

1. (a) $P_1 : \{s_1, s_4\}, \{s_2, s_3, s_5\}$
$(\nu(s_1,0) = s_4)E_1(\nu(s_4,0) = s_1)$ but $(\nu(s_1,1) = s_1)\not{E}_1(\nu(s_4,1) = s_3)$, so $s_1 \not{E}_2 s_4$.
$(\nu(s_2,1) = s_3)\not{E}_1(\nu(s_3,1) = s_4)$ so $s_2 \not{E}_2 s_3$.
$(\nu(s_2,0) = s_3)E_1(\nu(s_5,0) = s_3)$ and $(\nu(s_2,1) = s_3)E_1(\nu(s_5,1) = s_3)$ so $s_2 E_2 s_5$.
Since $s_2 \not{E}_2 S_3$ and $s_2 E_2 s_5$, it follows that $s_3 \not{E}_2 s_5$.
Hence P_2 is given by $P_2 : \{s_1\}, \{s_2, s_5\}, \{s_3,\}, \{s_4\}$. $(\nu(s_2,x) = s_3)E_2(\nu(s_5,x) = s_3)$ for $x = 0, 1$. Hence $s_2 E_3 s_5$ and $P_2 = P_3$.
Consequently, states s_2 and s_5 are equivalent.

(b) States s_2 and s_5 are equivalent.

(c) States s_2 and s_7 are equivalent; s_3 and s_4 are equivalent.

2. (a)



Consequently, 1100 is a distinguishing sequence since $\omega(s_1, 1100) = 0000 \neq 0001 = \omega(s_5, 1100)$.

(b) 100 (c) 00

3. (a) s_1 and s_7 are equivalent; s_4 and s_5 are equivalent.

(b) (i) 0000 (ii) 0 (iii) 00

M:	ν 0	ν 1	ω 0	ω 1
s_1	s_4	s_1	1	0
s_2	s_1	s_2	1	0
s_3	s_6	s_1	1	0
s_4	s_3	s_4	0	0
s_6	s_2	s_1	1	0

Supplementary Exercises

1. (a) False. Let $A = \{1,2\}, I = \{1,2\}, \mathcal{R}_1 = \{(1,1)\}, \mathcal{R}_2 = \{(2,2)\}$. Then $\cup_{i\in I}\mathcal{R}_i$ is reflexive but neither $\mathcal{R}_1$ nor $\mathcal{R}_2$ is reflexive. Conversely, however, if $\mathcal{R}_i$ is reflexive for all (actually at least one) $i \in I$, then $\cup_{i\in I}\mathcal{R}_i$ is reflexive.

(b) True. $\cap_{i\in I}\mathcal{R}_i$ reflexive $\Longleftrightarrow (a,a) \in \cap_{i\in I}\mathcal{R}_i$ for all $a \in A \Longleftrightarrow (a,a) \in \mathcal{R}_i$ for all $a \in A$ and all $i \in I \Longleftrightarrow \mathcal{R}_i$ is reflexive for all $i \in I$.

2. (i) (a) False. Let $A = \{1,2\}, \mathcal{R}_1 = \{(1,2)\}, \mathcal{R}_2 = \{(2,1)\}$. Then $\mathcal{R}_1 \cup \mathcal{R}_2$ is symmetric although neither $\mathcal{R}_1$ nor $\mathcal{R}_2$ is symmetric.

Conversely, however, if each $\mathcal{R}_i, i \in I$, is symmetric and $(x,y) \in \cup_{i\in I}\mathcal{R}_i$, then $(x,y) \in \mathcal{R}_i$ for some $i \in I$. Since $\mathcal{R}_i$ is symmetric, $(y,x) \in \mathcal{R}_i$, so $(y,x) \in \cup_{i\in I}\mathcal{R}_i$ and $\cup_{i\in I}\mathcal{R}_i$ is symmetric.

(b) If $(x,y) \in \cap_{i\in I}\mathcal{R}_i$, then $(x,y) \in \mathcal{R}_i$, for all $i \in I$. Since each $\mathcal{R}_i$ is symmetric, $(y,x) \in \mathcal{R}_i$, for all $i \in I$, so $(y,x) \in \cap_{i\in I}\mathcal{R}_i$ and $\cap_{i\in I}\mathcal{R}_i$ is symmetric.

The converse, however, is false. Let $A = \{1,2,3\}$, with $\mathcal{R}_1 = \{(1,2),(2,1),(1,3)\}$ and $\mathcal{R}_2 = \{(1,2),(2,1),(3,2)\}$. Then neither $\mathcal{R}_1$ nor $\mathcal{R}_2$ is symmetric, but $\mathcal{R}_1 \cap \mathcal{R}_2 = \{(1,2),(2,1)\}$ is symmetric.

(iii) (a) Let $A = \{1,2,3\}$ with $\mathcal{R}_1 = \{(1,2)\}$ and $\mathcal{R}_2 = \{(2,1)\}$. Then both $\mathcal{R}_1, \mathcal{R}_2$ are transitive but $\mathcal{R}_1 \cup \mathcal{R}_2$ is not transitive.

Conversely, for $A = \{1,2,3\}$ and $\mathcal{R}_1 = \{(1,3)\}, \mathcal{R}_2 = \{(1,2),(2,3)\}, \mathcal{R}_1 \cup \mathcal{R}_2 = \{(1,2),(2,3),(1,3)\}$ is transitive although $\mathcal{R}_2$ is not transitive.

(b) If $(x,y),(y,z) \in \cap_{i\in I}\mathcal{R}_i$, then $(x,y),(y,z) \in \mathcal{R}_i$ for all $i \in I$. With each $\mathcal{R}_i, i \in I$, transitive, it follows that $(x,z) \in \mathcal{R}_i$, so $(x,z) \in \cap_{i\in I}\mathcal{R}_i$ and $\cap_{i\in I}\mathcal{R}_i$ is transitive.

Conversely, however, $\{(1,2),(2,3)\} = \mathcal{R}_1$ and $\mathcal{R}_2 = \{(1,2)\}$ result in the transitive relation $\mathcal{R}_1 \cap \mathcal{R}_2 = \{(1,2)\}$ even though $\mathcal{R}_1$ is not transitive.

(ii) The results for part (ii) follow in a similar manner.

3. $(a,c) \in \mathcal{R}_2 \circ \mathcal{R}_1 \Longrightarrow$ for some $b \in A, (a,b) \in \mathcal{R}_2, (b,c) \in \mathcal{R}_1$. With $\mathcal{R}_1, \mathcal{R}_2$ symmetric, $(b,a) \in \mathcal{R}_2, (c,b) \in \mathcal{R}_1$, so $(c,a) \in \mathcal{R}_1 \circ \mathcal{R}_2 \subseteq \mathcal{R}_2 \circ \mathcal{R}_1$. $(c,a) \in \mathcal{R}_2 \circ \mathcal{R}_1 \Longrightarrow (c,d) \in \mathcal{R}_2, (d,a) \in \mathcal{R}_1$, for some $d \in A$. Then $(d,c) \in \mathcal{R}_2, (a,d) \in \mathcal{R}_1$ by symmetry, and $(a,c) \in$

$\mathcal{R}_1 \circ \mathcal{R}_2$, so $\mathcal{R}_2 \circ \mathcal{R}_1 \subseteq \mathcal{R}_1 \circ \mathcal{R}_2$ and the result follows.

4. (a) Reflexive, symmetric.

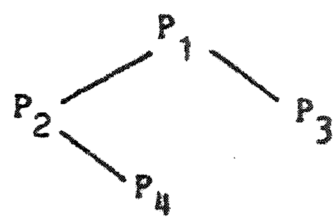
(b) Equivalence relation. Each equivalence class is of the form $A_r = \{t \in T |$ the area of $t = r, r \in \mathbf{R}^+\}$. Then $T = \cup_{r \in \mathbf{R}^+} A_r$.

(c) Reflexive, antisymmetric. (d) Symmetric.

(e) Equivalence relation. $[(1,1)] = \{(1,1),(2,2),(3,3),(4,4)\}$;
$[(1,2)] = \{(1,2),(2,1),(2,3),(3,2),(3,4),(4,3)\}$;
$[(1,3)] = \{(1,3),(3,1),(2,4),(4,2)\}; [(1,4)] = \{(1,4),(4,1)\}$.
$A = [(1,1)] \cup [(1,2)] \cup [(1,3)] \cup [(1,4)]$.

5. $(c,a) \in (\mathcal{R}_1 \circ \mathcal{R}_2)^c \iff (a,c) \in \mathcal{R}_1 \circ \mathcal{R}_2 \iff (a,b) \in \mathcal{R}_1, (b,c) \in \mathcal{R}_2$, for some $b \in B \iff (c,b) \in \mathcal{R}_2^c$, $(b,a) \in \mathcal{R}_1^c$, for some $b \in B \iff (c,a) \in \mathcal{R}_2^c \circ \mathcal{R}_1^c$.

6. (a) If P is a partition of A then $P \leq P$, so $\mathcal{R}$ is reflexive. For partitions P_i, P_j of A if $P_i \leq P_j$ and $P_j \leq P_i$, then $P_i = P_j$ and $\mathcal{R}$ is antisymmetric. Finally, if P_i, P_j, P_k are partitions of A and $P_i \mathcal{R} P_j, P_j \mathcal{R} P_k$, then $P_i \leq P_j$ and $P_j \leq P_k$, so each cell of P_i is contained in a cell of P_k and $P_i \leq P_k$. Hence $\mathcal{R}$ is transitive and is a partial order.
(b)



7. Let $\mathcal{U} = \{1,2,3,4,5\}, A = \mathcal{P}(\mathcal{U}) - \{\mathcal{U}, \emptyset\}$. Under the inclusion relation A is a poset with the five minimal elements $\{x\}, 1 \leq x \leq 5$, but no least element. Also, A has five maximal elements – the five subsets of $\mathcal{U}$ of size 4 – but no greatest element.

8. (b) $[(1,1)] = \{(1,1)\}$; $[(2,2)] = \{(1,4),(2,2),(4,1)\}$;
$[(3,2)] = \{(1,6),(2,3),(3,2),(6,1)\}$; $[(4,3)] = \{(2,6),(3,4),(4,3),(6,2)\}$.

9. $n = 10$

10. (a) For each $f \in \mathcal{F}, |f(n)| \leq 1|f(n)|$ for all $n \geq 1$, so $f\mathcal{R}f$, and $\mathcal{R}$ is reflexive. Second, if $f,g \in \mathcal{F}$, then $f\mathcal{R}g \implies (f \in O(g)$ and $g \in O(f)) \implies (g \in O(f)$ and $f \in O(g)) \implies g\mathcal{R}f$, so $\mathcal{R}$ is symmetric. Finally, let $f,g,h \in \mathcal{F}$ with $f\mathcal{R}g, g\mathcal{R}f, g\mathcal{R}h$, and $h\mathcal{R}g$. Then there exist $m_1, m_2 \in \mathbf{R}^+$, and $k_1, k_2 \in \mathbf{Z}^+$ so that $|f(n)| \leq m_1|g(n)|$ for all $n \geq k_1$, and $|g(n)| \leq m_2|h(n)|$ for all $n \geq k_2$. Consequently, for all $n \geq max\{k_1, k_2\}$ we have $|f(n)| \leq m_1|g(n)| \leq m_1 m_2 |h(n)|$ so $f \in O(h)$. And in a similar manner $h \in O(f)$. So $f\mathcal{R}h$ and $\mathcal{R}$ is transitive.

(b) For each $f \in \mathcal{F}$, f is dominated by itself, so $[f]S[f]$ and S is reflexive. Second, if $[g],[h] \in \mathcal{F}'$ with $[g]S[h]$ and $[h]S[g]$, then $g\mathcal{R}h$ (as in part (a)), and $[g] = [h]$. Consequently, S is antisymmetric. Finally, if $[f],[g],[h] \in \mathcal{F}'$ with $[f]S[g]$ and $[g]S[h]$, then f is dominated by g and g is dominated by h. So, as in part (a), f is dominated by h and $[f]S[h]$, making S transitive.

(c) Let $f, f_1, f_2 \in \mathcal{F}$ with $f(n) = n$, $f_1(n) = n+3$, and $f_2(n) = 2-n$. Then $(f_1+f_2)(n) = 5$, and $f_1 + f_2 \notin [f]$, because f is not dominated by $f_1 + f_2$.

11.

(a)

Adjacency List		Index List	
1	2	1	1
2	3	2	2
3	1	3	3
4	4	4	5
5	5	5	6
6	3	6	8
7	5		

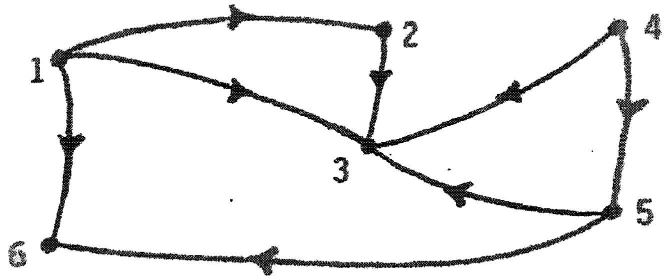
(b)

Adjacency List		Index List	
1	2	1	1
2	3	2	2
3	1	3	3
4	5	4	4
5	4	5	5
		6	6

(c)

Adjacency List		Index List	
1	2	1	1
2	3	2	2
3	1	3	3
4	4	4	6
5	5	5	7
6	1	6	8
7	4		

12.



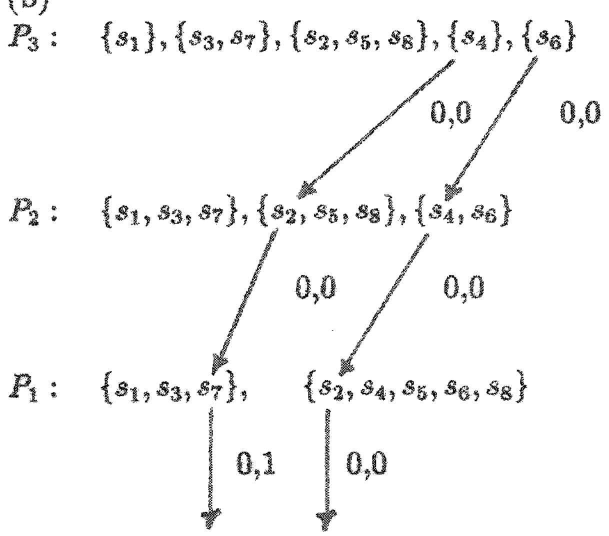
13. (a) For each $v \in V, v = v$ so $v\mathcal{R}v$. If $v\mathcal{R}w$ then there is a path from v to w. Since the graph G is undirected, the path from v to w is also a path from w to v, so $w\mathcal{R}v$ and $\mathcal{R}$ is symmetric. Finally, if $v\mathcal{R}w$ and $w\mathcal{R}x$, then a subset of the edges in the paths from v to w and w to x provide a path from v to x. Hence $\mathcal{R}$ is transitive and $\mathcal{R}$ is an equivalence relation.

(b) The cells of the partition are the (connected) components of G.

14. (a) $P_1 : \{s_1, s_3, s_7\}, \{s_2, s_4, s_5, s_6, s_8\}$
$P_2 : \{s_1, s_3, s_7\}, \{s_2, s_5, s_8\}, \{s_4, s_6\}$
$P_3 : \{s_1\}, \{s_3, s_7\}, \{s_2, s_5, s_8\}, \{s_4\}, \{s_6\}$
$P_4 = P_3$

	ν		ω	
M:	0	1	0	1
s_1	s_3	s_6	1	0
s_2	s_3	s_3	0	0
s_3	s_3	s_2	1	0
s_4	s_2	s_3	0	0
s_6	s_4	s_1	0	0

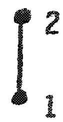
(b)



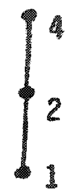
Hence $\omega(s_4, 000) = 001 \neq 000 = \omega(s_6, 000)$, so 000 is a distinguishing string for s_4 and s_6.

15. One possible order is 10, 3, 8, 6, 7, 9, 1, 4, 5, 2, where program 10 is run first and program 2 last.

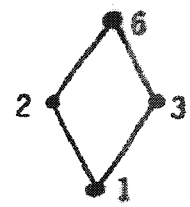
16. (a) (i) $n = 2$:



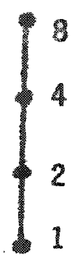
(ii) $n = 4$:



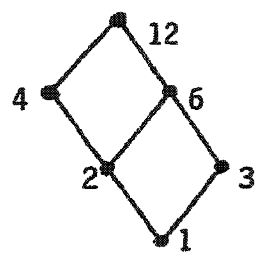
(iii) $n = 6$:



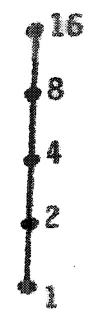
(iv) $n = 8$:



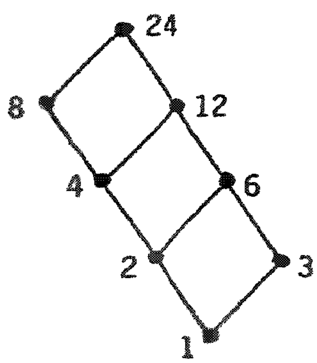
(v) $n = 12$:



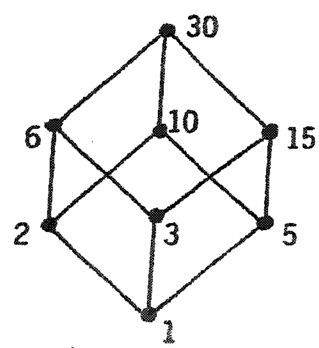
(vi) $n = 16$:



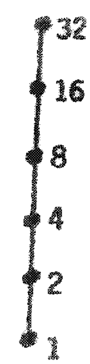
(viii) $n = 24$:



(viii) $n = 30$:



(ix) $n = 32$:



(b) For $2 \leq n \leq 35$, n can be written in one of the following nine forms: (i) p; (ii) p^2; (iii) pq; (iv) p^3; (v) p^2q; (vi) p^4; (vii) p^3q; (viii) pqr; (ix) p^5, where p, q, r denote distinct primes. The Hasse diagrams for these representations are given by the structures in part (a).

For $n = 36 = 2^2 \cdot 3^2$, we must introduce a new structure.

(c) The converse is false. $\tau(24) = 8 = \tau(30)$ but the Hasse diagrams in (vii) and (viii) of part (a) are not the same.

(d) This follows from the definitions of the gcd and lcm and the result of Example 4.45.

17. (b) $[(0.3, 0.7)] = \{(0.3, 0.7)\}$ $\quad [(0.5, 0)] = \{(0.5, 0)\}$ $\quad [(0.4, 1)] = \{(0.4, 1)\}$

$[(0,0.6)] = \{(0,0.6),(1,0.6)\}$ $\qquad$ $[(1,0.2)] = \{(0,0.2),(1,0.2)\}$
In general, if $0 < a < 1$, then $[(a,b)] = \{(a,b)\}$; otherwise, $[(0,b)] = \{(0,b),(1,b)\} = [(1,b)]$.
(c) The lateral surface of a cylinder of height 1 and base radius $1/2\pi$.

18. (a) If $C \subseteq \mathcal{U}$, then $0 \le |C| \le 3$. For $0 \le k \le 3$ there are $\binom{3}{k}$ subsets C of $\mathcal{U}$ where $|C| = k$; each such subset C determines 2^k subsets $B \subseteq C$. Hence the relation $\mathcal{R}$ contains $\binom{3}{0}2^0 + \binom{3}{1}2^1 + \binom{3}{2}2^2 + \binom{3}{3}2^3 = (1+2)^3 = 3^3 = 27$ ordered pairs.

(b) For $\mathcal{U} = \{1,2,3,4\}$ the number of ordered pairs in $\mathcal{R}$ is $\binom{4}{0}2^0 + \binom{4}{1}2^1 + \binom{4}{2}2^2 + \binom{4}{3}2^3 + \binom{4}{4}2^4 = (1+2)^4 = 3^4 = 81$.

(c) For $\mathcal{U} = \{1,2,3,\ldots,n\}$, where $n \ge 1$, there are 3^n ordered pairs in the relation $\mathcal{R}$.

19. Since $|\mathcal{U}| = n$, $|\mathcal{P}(\mathcal{U}| = 2^n$ and so there are $(2^n)(2^n) = 4^n$ ordered pairs of the form (A,B) where $A,B \subseteq \mathcal{U}$. From Exercise 18 (above) there are 3^n order pairs of the form (A,B) where $A \subseteq B$. [Note: If $(A,B) \in \mathcal{R}$, then so is (B,A).] Hence there are $3^n + 3^n - 2^n$ ordered pairs (A,B) where either $A \subseteq B$ or $B \subseteq A$, or both. We subtract 2^n because we have counted the 2^n ordered pairs (A,B), where $A = B$, twice. Therefore the number of ordered pairs in this relation is $4^n - (2 \cdot 3^n - 2^n) = 4^n - 2 \cdot 3^n + 2^n$.

20. (a) There are 2^m equivalence classes – one for each subset of B.
(b) 2^{n-m}

21. (a) (i) $B\mathcal{R}A\mathcal{R}C$; (ii) $B\mathcal{R}C\mathcal{R}F$
$B\mathcal{R}A\mathcal{R}C\mathcal{R}F$ is a maximal chain. There are six such maximal chains.

(b) Here 11 $\mathcal{R}$ 385 is a maximal chain of length 2, while 2 $\mathcal{R}$ 6 $\mathcal{R}$ 12 is one of length 3. The length of a longest chain for this poset is 3.
(c) (i) $\emptyset \subseteq \{1\} \subseteq \{1,2\} \subseteq \{1,2,3\} \subseteq \mathcal{U}$;
(ii) $\emptyset \subseteq \{2\} \subseteq \{2,3\} \subseteq \{1,2,3\} \subseteq \mathcal{U}$.
There are $4! = 24$ such maximal chains.
(d) $n!$

22. If c_1 is not a minimal element of $(A,\mathcal{R})$, then there is an element $a \in A$ with $a\mathcal{R}c_1$. But then this contradicts the maximality of the chain $(C,\mathcal{R}')$

The proof for c_n maximal in $(A,\mathcal{R})$ is similar.

23. Let $a_1\mathcal{R}a_2\mathcal{R}\ldots\mathcal{R}a_{n-1}\mathcal{R}a_n$ be a longest (maximal) chain in $(A,\mathcal{R})$. Then a_n is a maximal element in $(A,\mathcal{R})$ and $a_1\mathcal{R}a_2\mathcal{R}\ldots\mathcal{R}a_{n-1}$ is a maximal chain in $(B,\mathcal{R}')$. Hence the length of a longest chain in $(B,\mathcal{R}')$ is at least $n-1$. If there is a chain $b_1\mathcal{R}'b_2\mathcal{R}'\ldots\mathcal{R}'b_n$ in $(B,\mathcal{R}')$ of length n, then this is also a chain of length n in $(A,\mathcal{R})$. But then b_n must be a maximal element of $(A,\mathcal{R})$, and this contradicts $b_n \in B$.

24. (a) {2,3,5}; {5,6,7,11}; {2,3,5,7,11}

(b) $\{\{1,2\},\{3,4\}\}$, $\{\{1,2,3\}, \{2,3,4\}\}$; 4

(c) Consider the set M of all maximal elements in $(A, \mathcal{R})$. If this set is not an antichain then there are two elements $a, b \in M$ where $a\mathcal{R}b$ or $b\mathcal{R}a$. Assume, without loss of generality, that $a\mathcal{R}b$. If this is so, then a is *not* a maximal element of $(A, \mathcal{R})$. Hence $(M, (M \times M) \cap \mathcal{R})$ is an antichain in $(A, \mathcal{R})$.

The proof for the set of all minimal elements is similar.

25. If $n = 1$, then for all $x, y \in A$, if $x \neq y$ then $x\mathcal{R}y$ and $y\mathcal{R}x$. Hence $(A, \mathcal{R})$ is an antichain, and the result follows.

Now assume the result true for $n = k \geq 1$, and let $(A, \mathcal{R})$ be a poset where the length of a longest chain is $k+1$. If M is the set of all maximal elements in $(A, \mathcal{R})$, then $M \neq \emptyset$ and M is an antichain in $(A, \mathcal{R})$. Also, by virtue of Exercise 23 above, $(A - M, \mathcal{R}')$, for $\mathcal{R}' = ((A - M) \times (A - M)) \cap \mathcal{R}$, is a poset with k the length of a longest chain. So by the induction hypothesis $A - M = C_1 \cup C_2 \cup \ldots \cup C_k$, a partition into k antichains. Consequently, $A = C_1 \cup C_2 \cup \ldots \cup C_k \cup M$, a partition into $k+1$ antichains.

26. (a) Since $96 = 2^5 \cdot 3$, there are $\frac{1}{7}\binom{12}{6} = 132$ ways to totally order the partial order of 12 positive integer divisors of 96.

(b) Here we have $96 > 32$ and must now totally order the partial order of 10 positive integer divisors of 48. This can be done in $\frac{1}{6}\binom{10}{5} = 42$ ways.

(c) Aside from 1 and 3 there are ten other positive integer divisors of 96. The Hasse diagram for the partial order of these ten integers – namely, 2,4,6,8,12,16,24,32,48,96 – is structurally the same as the Hasse diagram for the partial order of positive integer divisors of 48. So as in part (b) the answer is 42 ways.

(d) Here there are 14 such total orders.

27. (a) There are n edges – namely, $(0,1), (1,2), (2,3), \ldots, (n-1, n)$.

(b) The number of partitions, as described here, equals the number of compositions of n. So the answer is 2^{n-1}.

(c) The number of such partitions is $2^{3-1} \cdot 2^{5-1} = 64$, for there are 2^{3-1} compositions of 3 and 2^{5-1} compositions of $5 (= 12 - 7)$.